# *Feasible High Growth Investment Strategy: growth optimal portfolios applied to Dow Jones stocks*

**Keywords**: *Growth Optimal Portfolios, Dow Jones Industrial Average Stocks, High Growth Equity Returns, Feasible Investment Strategy, Ridge Regression*

**Abstract**

A number of investment strategies designed to maximise portfolio growth are tested on a long-run US equity data set. The application of these growth optimal portfolio techniques produces impressive rates of growth, despite the fact that the assumptions of normality and stability that underlie the growth optimal model are shown to be inconsistent with the data.

Growth optimal portfolios are constructed by rebalancing the portfolio weights of Dow Jones Industrial Average (DJIA) stocks each month with the aim of maximising portfolio growth. These portfolios are shown to produce growth rates that are up to twice those of the benchmark, equally weighted, minimum variance and 15% drift portfolios. The key to the success of the classic, no short-sales, growth optimal portfolio strategy lies in its ability to select for portfolio inclusion a small number of (DJIA) stocks during their high growth periods.

The study introduces a variant of ridge regression to form the basis of one of the growth focussed investment strategies. The ridge growth optimal technique overcomes the problem of numerically unstable portfolio weights that dogs the formation of short-sales allowed growth portfolios. For the short-sales not allowed growth portfolio, the use of the ridge estimator produces increased asset diversification in the growth portfolio, while at the same time reducing the amount of portfolio adjustment required in rebalancing the growth portfolio from period to period.

B.F.Hunt

Graduate School of Business

University of Technology Sydney

Ben.hunt@uts.edu.au

## *Introduction*

The expected rate of growth in value is considered by many investors to be the pre-eminent characteristic of an investment portfolio. Ways to construct portfolios that maximise expected growth are well documented (for example, Hakansson (1971), Luenberger (1998) and Hunt (2002)). In addition Cower pioneered the application of elements of information theory to portfolio construction. Cower's so-called universal portfolio mimics constantly rebalanced portfolios that achieve optimal portfolio growth rates (See Cower (1991), Cower and Ordentlich (1996) Cross and Barron (2003))

Considering the importance of expected portfolio growth to both professional and retail investors, it is surprising that so few examples of studies that focus on the empirical strategies to maximise portfolio growth exist. This study aims to redress this deficiency by applying growth optimal techniques to long-run US equity data. As a first step in the study, let us set out a stochastic model of asset price evolution, upon which the growth optimising investment strategy will be based.

Suppose that the passage of an asset price, S, through time, t, is governed by geometric Brownian motion (generalised Weiner process):

$$dS(t) = \mu S(t)\,dt + \sigma S(t)\,dz \qquad (1)$$

where $\mu$ is the rate of drift and z is a standard Weiner process. The expected rate of growth of the asset, E[G], over time t, can, using Ito's lemma, be derived as:

$$G = E\left[\ln\left(\frac{S(t)}{S(0)}\right)\right] = \left(\mu - \frac{1}{2}\sigma^2\right)t \qquad (2)$$

The combinatorial properties of normal random variables dictate that if the value of n assets follows a geometric Brownian motion, the value of a combination of these assets, $S_p$, defined by a portfolio weights vector, $\mathbf{w}^T = (w_1, \ldots, w_n)$ will also be characterised by geometric Brownian motion:

$$dS_p(t) = \mu_p S_p(t)\,dt + \sigma_p\, S_p(t)\,dz \tag{3}$$

and will have an expected portfolio growth rate per unit of time, $g_p$, where:

$$\begin{aligned} g_p &= \frac{G_P}{t} \\ &= \left(\mu_p - \frac{1}{2}\sigma_p^2\right) \\ &= \left(\mathbf{w}^T\boldsymbol{\mu} - \frac{1}{2}\mathbf{w}^T\boldsymbol{\Omega}\,\mathbf{w}\right) \end{aligned} \tag{4}$$

where $\boldsymbol{\Omega}$ is an nxn matrix of variances and covariances, and $\boldsymbol{\mu}$ is a vector of individual expected drift rates per unit of time, $\boldsymbol{\mu}^T=(\mu_1, \ldots, \mu_n)$. The variance of the growth rate, $\sigma_p^2$ is:

$$\sigma_p^2 = \mathbf{w}^T\boldsymbol{\Omega}\,\mathbf{w}/t \tag{5}$$

It is evident from (4) that the rate of growth of a portfolio of assets is governed by the choice of the individual asset weightings, **w**. Naturally, the structure of w may be fashioned to maximise the expected rate of growth. The portfolio, w*, that maximises expected portfolio growth is referred to as the *growth optimal portfolio*.

A strategy designed to maximise expected growth has an obvious and intuitive appeal. Moreover, maximising expected growth has strong theoretical support. Consider the broad class of power utility of wealth functions:

$$U(W) = \frac{1}{\gamma}W^{\gamma} \tag{6}$$

The recursive nature of a utility function such as (6) means that the problem of maximisation of expected utility of wealth after n periods, $W_n$, reduces to a myopic strategy of the maximisation of wealth over one period, $W_1$. Further, if $\gamma$ is small, the expected value of power utility E[U(W)] is closely approximated by:

$$\begin{aligned} E[U(W_1)] &\approx E[\ln(W_1) + \frac{\gamma}{2}(\ln W_1)^2] \\ &\approx E[g] + \frac{\gamma}{2}(\sigma^2 + (E[g])^2] \end{aligned} \tag{7}$$

Thus, when $\gamma$ is small, it follows that the only two variables of interest in the quest to maximise expected utility of n period wealth, are the expected growth rate and the variance of the growth rate. Investors with a log utility function $U(W_n) = \ln(W_n)$, which is the limit case of (7) when $\gamma \to 0$, will choose between investments based solely on expected portfolio growth. Moreover, Luenberger (1993) provides a broader rationale for basing portfolio choice on expected growth using so-called tail preference theory.

Investment techniques based on optimising expected growth have appeal to both theorists and practitioners as they:

- ✓ are consistent with n period utility maximisation,
- ✓ suggest asset diversification,
- ✓ maximise expected terminal value of wealth and
- ✓ minimise the expected time required for accumulated wealth to reach any specified threshold value.

The theoretical attractiveness of maximal growth portfolios is clear. What is less apparent is whether or not investment strategies based on growth portfolios are efficacious. The aim of this paper is to examine the suitability of growth optimal portfolio techniques **to** the US equity investment environment.

## *Data*

Hakansson (1971) suggested that growth optimal portfolios dominate all other portfolios in the long run. While Merton and Samuleson (1974) pointed out the fallacy in this argument, it remains true that it is easier to identify the characteristics of alternative investment portfolios when observed over a long period of time. The desire to test the efficacy of growth-oriented investment led us to seek out a long-run US equity data set. The study applies the growth optimal portfolio investment strategy techniques to 25 years of monthly data, starting in May 1977 and ending in April 2002.

The data set comprise**d** dividend-adjusted price observations on the 30 companies in the Dow Jones Industrial Index (DJIA). While the growth investment strategies were applied to 25 years of monthly data, the study required a further 5 years of data in order to estimate the necessary drift and volatility parameters. Thus the full data set extended from April 1972 to April 2002.[1] Table 1 shows that only 21 of the 30 DJIA stocks had a full set of data. Nine DJIA stocks were included in the index after the commencement date of April 1972. The statistics quoted in Table 1 were estimated where possible over the strategy period May 1977 to April 2002. Otherwise the statistics were estimated over the period delineated by their starting date and April 2002.

The price data were transformed into measures of periodic growth using the continuous growth formula.

$$g_{i,t} = \ln(P_{i,t} / P_{i,t-1}) \qquad (8)$$

where $P_t$ is the dividend-adjusted price of asset i in month t and $g_{i,t}$ is the growth of asset i in month t.[2]

Table 1 displays annualised statistics on rates of growth, drift and volatility of growth for the 30 companies included in the data set. The annual rate of growth of asset i, $\hat{g}_i$, was estimated as the sample aggregate growth divided by the 25 years of the sample. The estimate of the asset drift rate, $\hat{\mu}_{i,}$, was computed as:

$$\hat{\mu}_{i,} = \hat{g}_i - \hat{\sigma}_i^2 / 2 \qquad (9)$$

Where $\hat{\sigma}_i^2$ is the estimate of the i[th] asset variance.

The DJIA stocks produced some impressive growth over the 25-year period. Wall-Mart, Microsoft, Home Deposit and Intel produced growth rates in excess of 20%pa over the 25n year period. At the other end of the growth spectrum, six stocks recorded growth rates lower than 10%pa. A benchmark portfolio of equally weighted DJIA stocks grew by 13.73%pa.

**Table 1: Dow Jones Stocks: Descriptive Statistics**

| *Symbol* | *Name* | *Start** | *Growth* | *Drift*† | *Volatility* |
|---|---|---|---|---|---|
| AA | Alcoa Inc. | | 12.40% | 16.74% | 29.43% |
| AXP | American Express Co. | Apr-77 | 14.14% | 18.28% | 28.79% |
| BA | Boeing Corp. | | 13.31% | 18.26% | 31.46% |
| C | Citigroup Inc | Jul-86 | 16.56% | 22.76% | 35.22% |
| CAT | Caterpilla Inc. | | 7.17% | 11.49% | 29.41% |
| DD | DuPont deNemours | | 11.91% | 14.70% | 23.61% |
| DIS | Walt Disney Co. | | 14.96% | 19.53% | 30.21% |
| EK | Eastman Kodak | | 6.00% | 9.29% | 25.64% |
| GE | General Electric Co. | | 13.44% | 15.82% | 21.81% |
| GM | General Motors | | 8.75% | 12.51% | 27.41% |
| HD | Home Deposit | Aug-84 | 28.23% | 33.38% | 32.09% |
| HON | Honeywell International Inc. | | 9.70% | 14.41% | 30.67% |
| HWP | Hewlett-Packard Co. | Aug-76 | 10.95% | 17.10% | 35.09% |
| IBM | International Business Machines | | 6.73% | 10.24% | 26.49% |
| INTC | Intel Corp. | Jul-86 | 26.82% | 36.98% | 45.07% |
| IP | International Paper Co. | | 8.28% | 12.30% | 28.35% |
| JNJ | Johnson & Johnson | Jan-77 | 17.44% | 20.02% | 22.69% |
| JPM | J.P.Morgan Chase & Co. | Dec-83 | 11.92% | 18.37% | 35.90% |
| KO | Coca-Cola Co. | | 17.48% | 19.97% | 22.35% |
| MCD | McDonalds Corp. | | 13.59% | 16.05% | 22.18% |
| MMM | Minnesota Mining & Manufacture | | 12.90% | 14.89% | 19.96% |
| MO | Phillip Morris | | 15.63% | 18.75% | 25.00% |
| MRK | Merk & Co. | | 17.16% | 19.91% | 23.46% |
| MSFT | Microsoft Corp. | Mar-86 | 34.42% | 42.64% | 40.56% |
| PG | Proctor & Gamble Co. | | 12.08% | 14.54% | 22.18% |
| SBC | SBC Communications Inc. | Jul-84 | 14.37% | 16.94% | 22.66% |
| T | AT&T Corp. | | 2.23% | 5.83% | 26.84% |
| UTX | United Technologies Corp. | | 14.01% | 17.92% | 27.98% |
| WMT | Wal-Mart Stores Inc. | | 28.92% | 33.13% | 29.02% |
| XOM | Exon Mobil Corp. | | 10.41% | 11.78% | 16.60% |
| Equal | Equally weighted average | | 13.75% | 15.10% | 16.45% |

* Unless otherwise indicated, the stock data set starts in April 1972. The 1972 start date was chosen to provide a 5-year estimation period prior to the 25 year strategy period. However, the reported statistics on growth, drift and volatility statistics are estimated over the strategy period May 1977 to April 2002.

† The stock drift constant (the $\mu$ of equation (1)) was computed for each stock as the rate of growth plus half the variance.

The drift statistic, $\mu_i$, for each stock, i, was calculated as the stock average growth, $g_i$ plus half the variance, $\sigma^2$. The figures in Table 1 show that on average the variance term added approximately 4%pa to the growth rates to produce the drift figures. The inflation of a stock's growth rate by a proportion of variance to produce a drift parameter, is at the heart of what Luenbuger ( 1998) calls "volatility pumping" investment strategies. Stocks with high volatility will tend to have a high implied growth rate. The inclusion of these high-drift/high-volatility stocks in a portfolio will have a synergistic effect, as the portfolio will capture the high-drift rates of the individual stocks while the high volatilities will be diversified away within in the portfolio.

The five DJIA stocks with the highest rates of drift over the 25-year period were Microsoft, Intel, Home Deposit, Wall-Mart and Citigroup. The five stocks with the lowest rates of drift were AT&T, Eastman Kodak, IBM, Caterpilla and Exon.

Table 2 and Figure 1 contain further evidence of the relationship amongst DJIA stock, growth, drift and volatility. Table 2 plots asset growth and drift against asset volatility for the 30 DJIA stocks. Visual inspection of Figure 1 indicates a positive relationship amongst growth and drift and volatility. Table 2 quantifies the relationship amongst the three variables of growth, drift and volatility for the 30 stocks via classical correlation and rank correlation statistics.

Figure 1: Growth and Drift versus Volatility

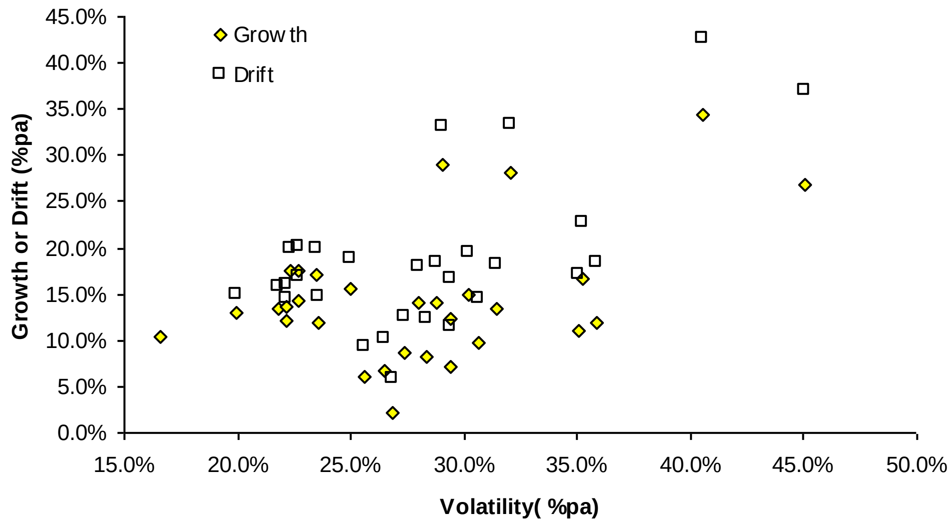



Table 2 shows that, as expected, the relationship between growth and volatility is positive. The strongest relationship amongst the three variables exists between growth and drift. The drift statistic for each asset is computed as a combination of the asset growth and volatility statistics. Asset growth contributes, on average, approximately 80% to the value of the drift statistic, so it is not surprising that growth and drift are highly correlated. The 20%, on average, contribution of volatility to the drift statistic accounts for the mid-range correlation and rank correlation between drift and volatility.

| **Table 2: Growth, Drift and Volatility Correlation** | | |
|---|---|---|
| | *Correlations* | |
| | Drift | Volatility |
| Growth | 0.9795 | 0.4455 |
| Drift | | 0.6147 |
| | | |
| | *Rank correlations* | |
| | Drift | Volatility |
| Growth | 0.9439 | 0.1671 |
| Drift | | 0.4211 |

## *Testing the Assumptions of the Growth Model*

There are a number of assumptions implicit in the model of growth upon which the investment strategies tested in this paper are based. Most obviously, the Weiner process of equation (1) assumes investment returns are normally distributed. Possibly less obvious is the model's reliance on the stability of stochastic process parameters of $\boldsymbol{\mu}$ and $\boldsymbol{\Omega}$. The degree to which these assumptions are consistent with the features of the historical data set ought to provide a guide to the likely success or otherwise of growth optimal investment strategies.

### Normality

The 30 companies and the benchmark equally weighted portfolio were tested for normality of returns with the results recorded in Table 3. Three tests of normality, based on skewness and kurtosis measures, were applied to the data set. The results of these tests reveal that the periodic growth rates were far from normally distributed.

Twenty of the 30 stocks displayed significant skewness at the 5% level at least. In addition, only four of the 30 stocks had returns that were not significantly leptokurtic. Predictably, the Jacque-Berra statistic, which jointly tests for skewness and kurtosis, rejected normality for the large majority of stocks (26 cases).

**Table 3: Tests of Normality[3] and Stability**

| Symbol | Skewness | | Kurtosis | | J-B | | ANOVA | Kruksal Wallis | Var Ratio |
|---|---|---|---|---|---|---|---|---|---|
| AA | 0.07 | | 2.45 | ** | 30.94 | ** | 0.19 | 1.48 | 3.52** |
| AXP | -0.88 | ** | 2.73 | ** | 72.85 | ** | 1.08 | 3.56 | 2.74* |
| BA | -0.24 | | 2.44 | ** | 33.37 | ** | 0.63 | 0.82 | 3.89** |
| C | -0.99 | ** | 3.11 | ** | 87.40 | ** | 1.26 | 1.69 | 2.31 |
| CAT | -0.43 | ** | 1.83 | ** | 31.99 | ** | 0.81 | 5.28 | 1.82 |
| DD | -0.28 | * | 0.57 | * | 10.96 | ** | 1.30 | 3.49 | 2.33 |
| DIS | -0.39 | ** | 1.67 | ** | 28.41 | ** | 0.80 | 3.51 | 3.12* |
| EK | -0.95 | ** | 4.48 | ** | 101.54 | ** | 1.65 | 4.04 | 2.72* |
| GE | -0.19 | | 0.85 | ** | 12.45 | ** | 0.97 | 5.07 | 3.11* |
| GM | -0.48 | ** | 1.92 | ** | 35.33 | ** | 0.39 | 1.48 | 3.36** |
| HD | -0.40 | ** | 0.91 | ** | 19.28 | ** | 1.77 | 6.02 | 2.42* |
| HON | -0.78 | ** | 5.10 | ** | 94.12 | ** | 0.50 | 1.06 | 6.32** |
| HWP | -0.35 | * | 1.28 | ** | 21.99 | ** | 0.64 | 1.78 | 3.22* |
| IBM | -0.09 | | 1.04 | ** | 13.33 | ** | 1.07 | 4.49 | 4.01** |
| INTC | -0.76 | ** | 2.33 | ** | 58.39 | ** | 1.48 | 1.65 | 2.73* |
| IP | -0.14 | | 1.05 | ** | 14.10 | ** | 0.65 | 2.93 | 3.51** |
| JNJ | -0.27 | | 0.10 | | 4.70 | | 0.05 | 0.12 | 1.58 |
| JPM | -0.75 | ** | 2.31 | ** | 56.88 | ** | 0.36 | 0.37 | 4.89** |
| KO | -0.39 | ** | 1.08 | ** | 21.11 | ** | 2.60* | 9.80* | 3.68** |
| MCD | -0.28 | * | -0.19 | | 1.57 | | 0.84 | 2.53 | 2.25 |
| MMM | -0.42 | ** | 4.14 | ** | 60.76 | ** | 0.52 | 3.46 | 2.72* |
| MO | -0.52 | ** | 1.94 | ** | 37.83 | ** | 0.69 | 2.21 | 2.47* |
| MRK | -0.22 | | 0.25 | | 5.55 | | 1.04 | 4.73 | 2.40* |
| MSFT | -0.07 | | 1.18 | ** | 14.98 | ** | 0.80 | 2.08 | 3.09* |
| PG | -1.21 | ** | 7.49 | ** | 166.50 | ** | 0.54 | 3.93 | 3.70** |
| SBC | -0.37 | ** | 0.82 | ** | 17.12 | ** | 0.45 | 1.11 | 4.42** |
| T | -0.49 | ** | 3.70 | ** | 58.44 | ** | 0.50 | 2.77 | 12.97** |
| UTX | -1.35 | ** | 6.72 | ** | 175.07 | ** | 0.49 | 3.22 | 4.72** |
| WMT | -0.16 | | 0.96 | ** | 13.18 | ** | 1.74 | 6.12 | 2.31 |
| XOM | 0.13 | | 0.48 | | 6.80 | | 1.15 | 3.47 | 2.33 |
| Equal | -0.91 | ** | 4.65 | ** | 99.83 | ** | 0.98 | 3.80 | 4.01** |

* Indicates significance at the 5% level, ** indicates significance at the 1% level
[†] The kurtosis figure displayed was computed using Excel's KURT() function and is equal to the traditional measure of kurtosis less 3.

The results of the analysis of skewness and kurtosis allow us to confidently conclude that the data upon which we are to test the growth optimal portfolio strategies are largely non-normal. Exactly how the non-normality will impinge upon the investment results is problematical, as the effect that skewness and excess kurtosis will have on the growth of portfolios designed under an assumption of normality is not immediately apparent. The question of the stability of the stock's distributional statistics over time is perhaps more important than the actual values of those statistics.

## Serial Stability

The expected growth rate for each stock, the variance of that growth rate and the covariances between each stock's growth rate are essential inputs to the process of determining growth optimal portfolio weights. The growth optimal strategies implemented in this study rely on the stability of the input parameter estimates. Thus any serial instability in these input parameters will imperil the success of any investment strategy based on an assumption of parameter constancy.

**Figure 2: Equally weighted portfolio: sub-period growth, drift and volatility**

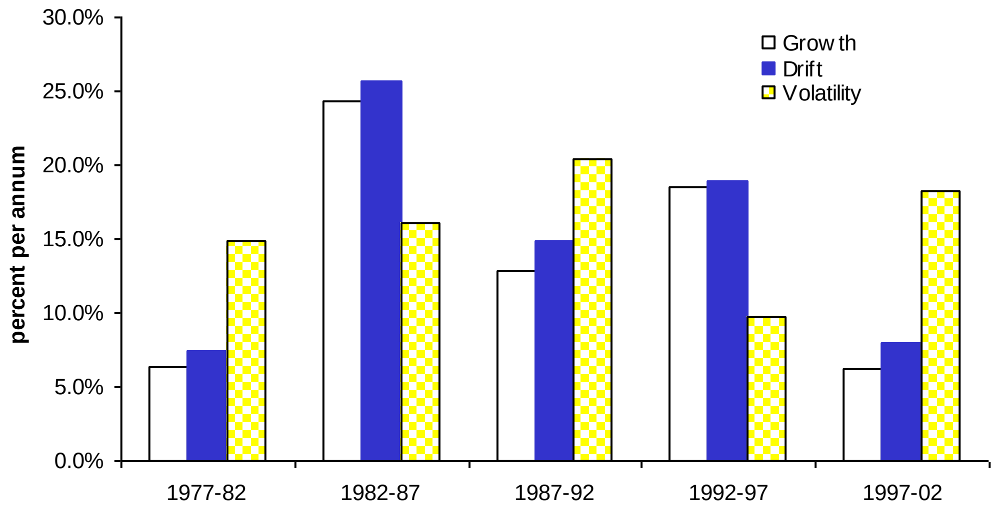



Figure 2 plots estimates of the average growth, drift and volatility, for a portfolio of equally weighted DJIA stocks over five equal sub-periods that make up the overall data-set period.

Casual analysis of the range of sub-period growth estimates suggests parameter instability. Sub-period growth ranges from a low of 6.45% in 1977-82 to a high of 24.47%pa in 1982-87. However, the result of applying formal tests for instability of the mean of growth for the individual stocks does not lead to the conclusion that these are unstable. Analysis of variances indicates instability in only one of the stocks, KO.

The Analysis of Variance results accord with the results of the Kruksal-Wallis test, which is the more suitable test given the non-normality of the data. The hypothesis of constancy of growth rates in all five sub-periods has been rejected for only KO.

The proposition that variance of growth rates is identical in each of the 5 sub-periods, was checked by using Hartley's test for homogeneity of variance (see Berenson and Levine (1992)). The ratio of the largest sub-period variance to the smallest sub-period variance, which is the key statistic in Hartley's test, is displayed in Table 2. Hartley's test indicates that the presence of serial instability of variance of growth rates exists in many of the sample stocks. The null hypothesis of equality of sub-period was rejected, at the 5% level at least, for 23 of the 30 companies.

While the variance for each stock is an input variable used to compute growth optimal portfolios, it is, however, the full covariance matrix that is the essential input item, and the variances represent only a small proportion of the larger covariance matrix. However, the preceding evidence of variance instability justified further research to ascertain whether the variance instability was also mirrored in covariance instability.

A test of the hypothesis of equality of sub-period covariance matrices employs the Box's M statistic, where:

$$M = n\sum_{i=1}^{m} \ln|\mathbf{\Omega}| - n_s \sum_{i=1}^{m} \ln|\mathbf{\Omega}_i| \qquad (10)$$

where m is the number of sub-periods; $n = m\,n_s$ is number of observations in the full sample; $n_s$ is the number of observations in each sub-period; $|\Omega|$ is the determinant of the overall, p-dimensioned, covariance matrices; and $|\Omega_i|$ is the determinant of the $i^{th}$ sub-period covariance matrix. Pearson (1969) shows that for large p, M is distributed as $F_{f1,f2}$ .[4] The sample M/b was computed as 3.58. This is to be compared to the 1% critical F of 1.07. Hence, it must be concluded that the sample data covariance matrix is not stationary.

The preceding results do not allow much scope for optimism as to the successful application of an investment technique based on growth optimal portfolios. This technique relies on an assumption of normality of period-by-period growth rates, as well as an implicit assumption of the stability of the distributional parameters contained within the expected growth rates and the covariance matrix of growth rates. Contrary to these assumptions, the analysis has shown that the DJIA stocks' data are leptokurtic and somewhat skewed, and are characterised by a non-stationary covariance matrix. However, the facts of the situation notwithstanding, we proceeded to test the efficacy of growth optimal portfolio investment techniques applied to the DJIA stock data.

## Application of Growth Optimal Portfolio Investment Techniques

This paper attempts to test a simple, practical investment strategy based on portfolios selected to have maximum expected growth rate. Testing any proposed investment strategy on the historical data involved stepping through each of the 300 monthly observations on the return of the DJIA companies. At any period, k, the following steps are undertaken:

1. The data on the previous n periods are employed to provide estimates of the expected drift rate, $\mu_i$, for each DJIA stock in the sample and to estimate each element of the covariance matrix, $\sigma_{i,j}$.
2. The estimates of the vector of expected drift rates and covariance matrix estimates are used to produce growth optimal portfolio weights, $w_k$.
3. The return on this portfolio in the next, ie k+1, period is computed.
4. The time-frame is moved forward one observation.

Steps 1 to 4 are repeated until the data set is exhausted.

The study uses the maximum number of DJIA stocks available at any point in time. Twenty-one stocks were part of the index as at May 1972. Thus, these stocks are included in the study for each of the 300 monthly periods. The remaining stocks are included in the study as soon as is permitted by the available data. The exact date of the addition of any one of the remaining nine stocks depended on the date of its inclusion in the DJIA and upon the length of the estimation period.

## Short-sales Allowed Portfolios

The structure of short-sales allowed, growth optimal portfolios is extensively explored in Hunt (2002). The GOP weights, **w***, vector has the following structure:

$$\mathbf{w}^* = \mathbf{A}\boldsymbol{\mu} + \mathbf{b} \qquad (12)$$

where:

$$\mathbf{A} = \boldsymbol{\Omega}^{-1}\left(\mathbf{I} - \frac{\boldsymbol{\iota}\boldsymbol{\iota}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}}{\boldsymbol{\iota}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\boldsymbol{\iota}}\right) \text{ and } \mathbf{b} = \frac{\boldsymbol{\Omega}^{-1}\boldsymbol{\iota}}{\boldsymbol{\iota}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\boldsymbol{\iota}}$$

It is our aim to proceed through the historical data set, estimating **r** and $\boldsymbol{\Omega}$, using these estimates to calculate the growth optimal weights, **w***, and to use these weights to produce a set of one-step-ahead returns for each of the 300 observations in the data set. The success or failure of the growth optimal investment techniques will be judged on the nature of the one-step-ahead returns produced by the strategy. The returns on three alternative investment strategies will provide a base against which to measure the growth optimal techniques.

These benchmark portfolios are:

1. the equally weighted portfolio,
2. the minimum variance portfolio and

3. the portfolio with an expected drift of 15%pa.

The equally weighted portfolio is a simple passive investment strategy and represents the absolute minimum “bar” against which alternatives ought to be measured. Note that performance of the equally weighted portfolio varies slightly with the length of the estimation period, as this determines exactly when the latest company additions to the DJIA become available for inclusion in the investment strategies.

The minimum variance point (MVP) strategy aims to minimise portfolio variance regardless of the expected level of portfolio drift. Weights for the minimum variance portfolio (MVP), $w_{MVP}$ are given by:

$$\mathbf{w}_{MVP} = \mathbf{b} = \frac{\mathbf{\Omega}^{-1}\boldsymbol{\iota}}{\boldsymbol{\iota}^{T}\mathbf{\Omega}^{-1}\boldsymbol{\iota}} \quad \textbf{(13)}$$

The final benchmark portfolio is one with an expected drift rate of 15%pa. The figure of 15%, while being arbitrary, is consistent with the historical record and is in general accord with investors’ expectations of share market returns over a long period.

The results arising from the application of the short-sales allowed growth optimal strategy, and application of the benchmark strategies, for parameter estimation period lengths of 3, 4 and 5 years, to the 25-year DJIA data set, are set out in Table 4 under the heading “Classical” strategies.

**Table 4: Short-sales Allowed Portfolio Strategy Returns**

| | Equally weighted | "Classic" strategies | | | Ridge constant = 0.05 | | |
|---|---|---|---|---|---|---|---|
| | | *MVP* | *15% drift* | *Growth* | *MVP* | *15% drift* | *Growth* |
| **Estimation period = 3 years** | | | | | | | |
| Average | 13.64% | -0.30% | 3.60% | 1158.72% | 12.40% | 12.41% | 18.40% |
| Volatility | 16.22% | 25.17% | 23.82% | 5272.68% | 15.39% | 15.40% | 33.98% |
| Length | 1.00 | 6.72 | 6.38 | 1027.54 | 1.04 | 1.18 | 4.15 |
| **Estimation period = 4 years** | | | | | | | |
| Average | 13.51% | 4.79% | 5.59% | 68.51% | 12.58% | 12.36% | 20.01% |
| Volatility | 16.27% | 18.09% | 18.15% | 1121.94% | 15.42% | 15.45% | 37.32% |
| Length | 1.00 | 4.95 | 4.84 | 236.59 | 1.05 | 1.18 | 4.58 |
| **Estimation period = 5 years** | | | | | | | |
| Average | 13.21% | 6.52% | 6.66% | -88.48% | 12.54% | 12.75% | 18.36% |
| Volatility | 16.16% | 17.26% | 17.01% | 613.01% | 15.42% | 15.52% | 43.07% |
| Length | 1.00 | 4.34 | 4.36 | 136.79 | 1.05 | 1.18 | 5.22 |

The first notable result is the excessive volatility associated with the short-sales allowed "classic" growth optimal portfolio. For example, the 3-year estimation period growth strategy, while exhibiting a growth rate in excess of 1000%pa, had a volatility in excess of 5000%pa. An examination of the growth optimal weights reveals that the short-sales allowed growth portfolios have gearing ratios that any investor would find impractically high. The strategy routinely required an asset to be short-sold more than 1000%. So-called portfolio length provides a measure of the extent of gearing within a portfolio that permits short-selling. Portfolio length, l, is defined as:

$$l = \sqrt{(n\,\mathbf{w}^T\mathbf{w})} \qquad \textbf{(14)}$$

An appreciation of the extent of gearing within a portfolio may be obtained by a comparison of the portfolio length of the least geared portfolio, the equally weighted portfolio, which has a length of unity. The classic short-sales allowed growth strategies had average portfolio lengths that ranged from 136.79 to 1027.54 depending on the length of the estimation period.

Not only were the individual growth portfolio asset weights absolutely large, they also gyrated wildly from period to period. The highly-geared portfolios were naturally characterised by high volatility of growth rates. The root cause of the volatility and instability of the short-sales allowed growth portfolios lay in the near singularity of the covariance matrix, $\mathbf{\Omega}$.

The inverse of the covariance matrix, $\mathbf{\Omega}^{-1}$, is necessary for the determination of the weights of the growth optimal portfolio, the MVP and the 15% drift portfolio. Unfortunately, a problem arises in the computation of $\mathbf{\Omega}^{-1}$ due to the multi-collinear nature of the periodic stock growth rates. The empirical estimates of the covariance, $\mathbf{\Omega}$, is at times close to being singular. The near singularity of $\mathbf{\Omega}$ results in the inverse being extremely sensitive to observations, and this in turn results in estimates of individual stock weights, $\mathbf{w}^*_i$, that gyrate wildly from observation to observation.

It is worth noting that the individual asset weights in an minimum variance portfolio can be interpreted as restricted least squares coefficients estimates. Now the ridge regression technique has long been employed to provide a solution to multi-collinearity in regression analysis.[5] The use of a ridge regression in portfolio construction can also be viewed from a Bayesian perspective. It is equivalent to employing a prior distribution of equal asset weights with constant variance. Our replacement of $\mathbf{\Omega}$ with an amended covariance matrix, $\mathbf{\Omega}_+$, where:

$$\Omega_+ = \Omega + d\,I \text{, d is a scalar, and I is the identity matrix} \qquad \textbf{(15)}$$

in the process of forming optimal portfolio weights, provides a solution to the multi-collinearity problem similar to that of ridge regression.

The use of a non-zero d in equation **(15)** produces “biased” estimates of the growth optimal portfolio and the benchmark portfolios. As the pivot, d, increases, the ridge estimate of growth optimal portfolio weights, $\mathbf{w}_+^*$, is biased away from the classic growth optimal portfolio weights towards those of the equally weighted portfolio. That is, in the limit, $w_+^* = 1/n$., where n is the number of assets in the set. In other words, the ridge estimator produces weights that are a combination of the classic estimator weights and those of the equally weighted portfolio.

A decision to use a ridge estimator necessarily requires a particular value for d. Unfortunately, there is no commonly accepted method for objectively selecting a value for d. A common approach in ridge regression analysis is to choose a value for d that provides "stabilised" estimates of the system parameters. After some experimentation, a figure of 5% was settled upon as a suitable value for d. Table 4 shows that the use of d = 0.05 reduced the average length of the of weights vector by between 95% and 99% depending on the length of the estimation period.

The results of applying the MVP, the 15% drift and the growth optimal strategy, with the ridge constant set to 0.05, for the three parameter estimation period lengths, are set out in Table 4. The bias of the ridge strategies towards the equally weighted strategy is immediately apparent. The MVP and 15% drift portfolios exhibit growth rates and volatility similar to those of the equally weighted portfolios when d is set at 5%. The equally weighted portfolio, the MVP portfolio and the 15% drift portfolio each produced a rate of growth of a little above 12.5%p.a., with an associated volatility of about 15.5% regardless of the length of the estimation period. In fact the performance of all four strategies, including the growth strategy, appears to be relatively independent of the length of the estimation period for the ridge portfolios.

The short-sales allowed growth portfolios, whose weights were estimated with a ridge factor of 0.05, were much better behaved than their zero ridge factor counterparts. The growth strategy portfolios outperformed the other benchmarks by more or less than 7%pa depending on the length of the estimation period.

**Figure 5: Short -sales Allowed Aggregate Portfolio Growth**

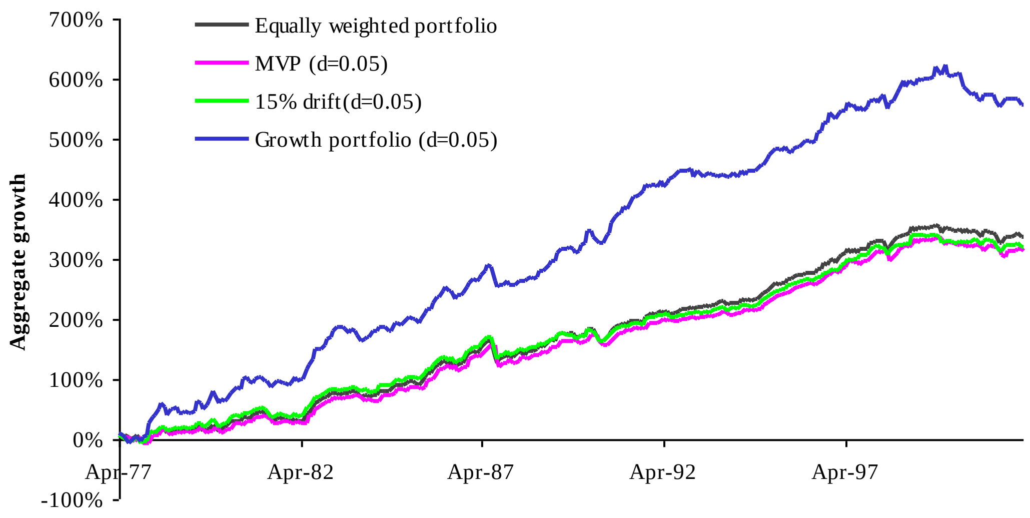



Figure 5 shows the extent of the superior performance by the short-sales allowed, ridge constant = 0.05, 4-year estimation period, portfolio over the 25 years of the data set. It should, however, be noted that high growth has come at the cost of high volatility. The volatility of the ridge growth portfolio was more than double that of the benchmark portfolios.

The short-sales allowed growth portfolio strategy results, while of interest, are largely academic, as the presence of short-sold shares in the portfolios of either professional or retail investors is not typical. An analysis of the results of growth portfolios where short-selling is not allowed will provide a more practical test of the growth investment strategy.

## Short-sales not allowed growth portfolios

While the short-selling of stock in most equity markets, including US, is allowed, it is not typical. Trialling growth optimal portfolios where a no short-sales restriction is imposed on portfolio weights, is a test of a more realistic strategy. The results from testing no short-sales growth portfolios are set out in Table 5.

The no short-sales growth portfolio performances are impressive. The statistics recorded in Table 5 show that the classic no short-sales, non-ridge (ie d=0), estimator produces portfolio growth rates in excess of 20%pa and up to 31%pa, depending on the length of the input estimation period.

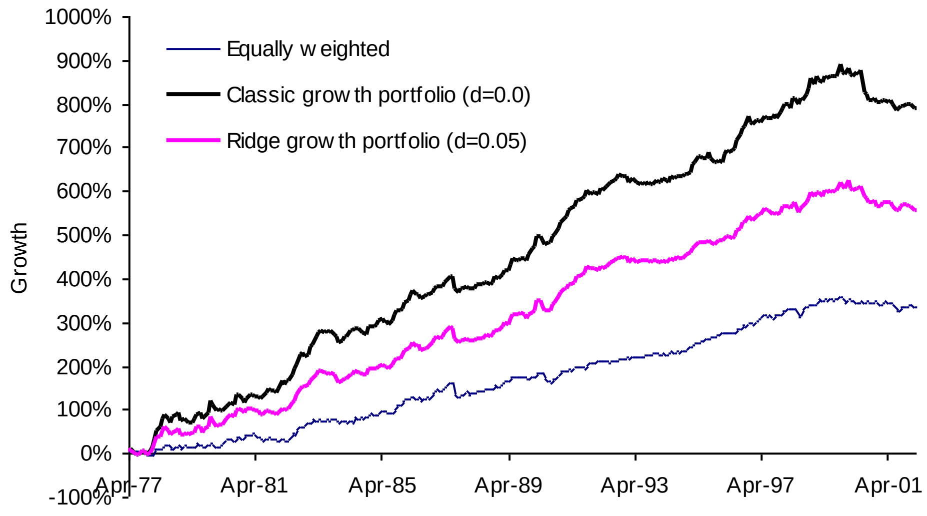
**Table 5: No Short-sales Allowed Portfolio Strategy Growth Rates**

| | **Equally weighted** | **"Classic" (d=0.00) strategies** | | | **Ridge (d=0.05)** | | |
|---|---|---|---|---|---|---|---|
| | | *MVP* | *15% drift* | *Growth* | *MVP* | *15% drift* | *Growth* |
| **Estimation period = 3 years** | | | | | | | |
| Average | 13.64% | 10.33% | 11.17% | 25.27% | 12.54% | 12.93% | 21.56% |
| Volatility | 16.22% | 13.90% | 14.43% | 30.93% | 15.42% | 15.63% | 24.51% |
| **Estimation period = 4 years** | | | | | | | |
| Average | 13.51% | 10.09% | 10.98% | 31.54% | 12.58% | 12.85% | 22.27% |
| Volatility | 16.27% | 13.57% | 13.98% | 31.61% | 15.42% | 15.72% | 24.19% |
| **Estimation period = 5 years** | | | | | | | |
| Average | 13.21% | 9.56% | 12.10% | 26.53% | 12.40% | 13.94% | 20.62% |
| Volatility | 16.16% | 13.71% | 14.30% | 32.38% | 15.39% | 15.63% | 23.79% |

The aggregate growth for the 2-year, 3-year and 4-year estimation period growth portfolios is depicted in Figure 6. The extent to which the growth portfolios outpaced the benchmark portfolios is clearly evident. The growth portfolios' performance is even more impressive when stated in dollar terms. One dollar invested in the no short-sales, 3-year, 4-year and 5-year estimation period, growth optimal portfolio strategy in April 1977 would have returned \$553.89, \$2,657.89 and \$759.59 respectively at the end of April 2002. These figures grossly exceed the returns on the equally weighted portfolios for the 3-year, 4-year and 5-year estimation periods of \$30.25, \$29.29 and \$27.19 respectively.

Figure 6 charts the accumulation of growth for the classic and the ridge 4-year estimation period, short-sales not allowed, growth investment strategies over the 25-year period. It is clearly apparent in Figure 6 that the classic strategy consistently produces accumulated growth which is more than double that of the benchmark equally weighted portfolio. It is equally apparent from Figure 6 that the ridge growth strategy, d=(0.05), produces a rate of growth that falls between those of the classic growth strategy and the equally weighted portfolio.

**Figure 6: Short-sales Not Allowed Aggregate Growth Rates**




The impressive performance of the classic, no short-sales growth strategy begs further analysis. It is clear from Table 5 that the growth-oriented strategies, while having considerably higher growth rates than the benchmark strategies, also have much higher volatility than the benchmark MVP, 15% growth and equally weighted strategies. Regardless of the length of the estimation period, the direct relationship between growth and volatility is clearly evident as Table 5 shows. The evidence shows that no single investment strategy clearly dominates any other strategy. Indeed, the results of this study provide strong support for what Luenberger (1998) calls the log mean-variance model.

Low-growth portfolios are associated with low volatility and high-growth portfolios are associated with high volatility. The point is, however, that regardless of the cost in terms of volatility, the portfolios designed for maximal growth did produce quite remarkable rates of growth. It is worth investigating the source of this growth.

**Table 6: No Short-sales, 4-year Estimation Period, Portfolio Properties.**

| Input estimation period | Portfolio type | Average growth rate | Volatility of growth rate | Average portfolio length | Average no of included assets | Average turnover of assets |
|---|---|---|---|---|---|---|
| | Equal weights | 13.51% | 16.27% | 1.00 | 26.58 | 0.43% |
| Classic no short-sales growth portfolios (d=0.00) | MVP | 10.09% | 13.57% | 2.57 | 9.04 | 9.98% |
| | 15% drift | 10.98% | 13.98% | 2.43 | 8.90 | 13.28% |
| | Growth | 31.54% | 31.61% | 4.70 | 1.56 | 13.29% |
| Ridge no short-sales growth portfolios (d=0.05) | MVP | 12.58% | 15.42% | 1.05 | 26.64 | 4.25% |
| | 15% drift | 12.85% | 15.72% | 1.21 | 23.92 | 9.91% |
| | Growth | 22.27% | 24.19% | 2.59 | 7.74 | 12.54% |

From here on, the analysis of results is restricted, for the sake of brevity, to the 4-year estimation period.

It is insightful to examine the average portfolio length and average number of included assets in the no short-sales growth optimal portfolios. The length of a no short-sales allowed portfolio is inversely indicative of its “diversity”. The length of a no short-sales, 25-stock portfolio can take values between one and five. The maximum portfolio length of 5 is achieved, for any 25-stock portfolio, when 100% of portfolio value is held in a single stock. At the other end of the spectrum is the maximally diversified, equally weighted portfolio with unit length.

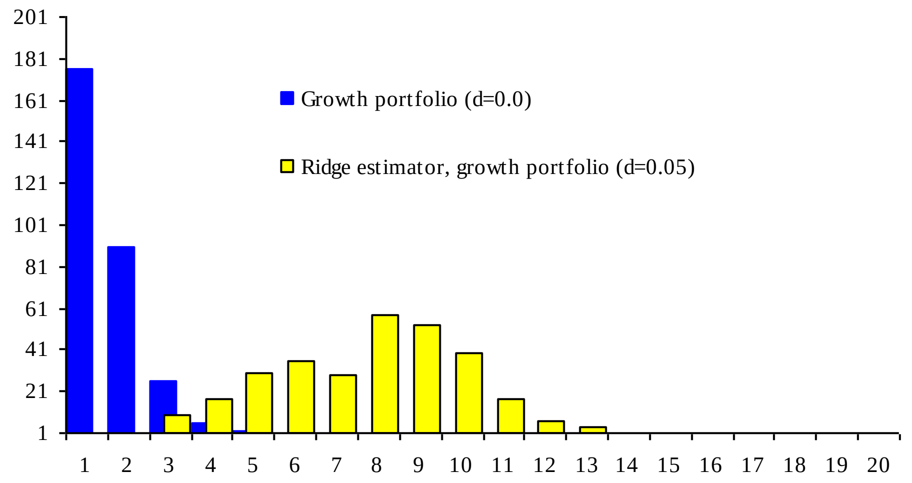
Table 6 shows that the “classic” no short-sales allowed, growth optimal portfolio with an average length of 4.70, is at the lower end of the diversity spectrum. Moreover, Table 6 shows that this portfolio contains on average only 1.56 assets in each period. Further, Figure 7, which shows the distribution of the number of assets held in each period, reveals that for the majority of the 300 monthly periods, the no short-sales growth optimal portfolio consisted of a single asset.

| Table 7: Growth Portfolio Disaggregated Growth | | | | | | |
|---|---|---|---|---|---|---|
| *Stock* | *Classic d=0.00)* | | | *Ridge d=0.05* | | |
| | No. of inclusions | Growth while included (%pa) | Growth while excluded | No. of inclusions | Growth while included | Growth while excluded |
| AA | 4 | -47.5% | 13.2% | 50 | 3.2% | 14.2% |
| AXP | 1 | -105.8% | 14.5% | 118 | 18.1% | 11.6% |
| BA | 44 | 27.8% | 10.8% | 97 | 9.7% | 15.1% |
| C | 12 | -25.0% | 18.3% | 109 | 22.6% | 13.1% |
| CAT | 2 | 32.9% | 7.0% | 47 | 13.7% | 6.0% |
| DD | 0 | | 11.9% | 33 | -14.2% | 15.1% |
| DIS | 33 | 37.9% | 12.1% | 76 | 27.7% | 10.6% |
| EK | 0 | | 6.0% | 33 | 1.1% | 6.6% |
| GE | 0 | | 13.4% | 66 | 11.8% | 13.9% |
| GM | 0 | | 8.7% | 40 | 6.8% | 9.0% |
| HD | 82 | 27.4% | 28.6% | 123 | 26.6% | 29.4% |
| HON | 2 | 27.3% | 9.6% | 60 | 7.5% | 10.2% |
| HWP | 5 | 33.8% | 10.6% | 95 | 13.3% | 9.8% |
| IBM | 5 | 7.2% | 6.7% | 90 | 13.4% | 3.9% |
| INTC | 58 | 21.3% | 28.1% | 118 | 23.4% | 29.0% |
| IP | 0 | | 8.3% | 12 | -42.6% | 10.4% |
| JNJ | 0 | | 17.4% | 79 | 16.7% | 17.7% |
| JPM | 5 | 4.2% | 12.1% | 59 | 23.8% | 9.0% |
| KO | 0 | | 17.5% | 154 | 21.2% | 13.5% |
| MCD | 0 | | 13.6% | 77 | 19.1% | 11.7% |
| MMM | 0 | | 12.9% | 3 | -12.8% | 13.2% |
| MO | 0 | | 15.6% | 117 | 18.6% | 13.7% |
| MRK | 28 | 19.3% | 16.9% | 115 | 11.9% | 20.4% |
| MSFT | 75 | 15.3% | 40.8% | 145 | 28.4% | 40.1% |
| PG | 0 | | 12.1% | 1 | 29.0% | 12.0% |
| SBC | 0 | | 14.4% | 25 | 1.2% | 15.6% |
| T | 0 | | 2.2% | 2 | -42.9% | 2.5% |
| UTX | 8 | -1.3% | 14.4% | 109 | 3.1% | 20.2% |
| WMT | 104 | 30.3% | 28.2% | 241 | 29.6% | 26.0% |
| XOM | 0 | | 10.4% | 27 | 10.9% | 10.4% |
| The statistics in the table are for stocks included in no short-sales, growth optimal portfolio strategies, employing a 4 year estimation period | | | | | | |

Table 7 sets out statistics relating to included stocks in a growth optimal strategy over 25 years. Only 16 of the 30 stocks were ever included in the classic growth strategy. Of these stocks only 9 were included for 12 months or more. Home Deposit, Intel, Microsoft and Walmart were included in the classic growth portfolio for 50 months or more. The ridge growth strategy was more inclusive of assets. Twenty of the 30 assets were included for 50 months or more.

Table 7 sets out figures for each asset's growth, separated into the growth generated while the asset was included in growth portfolios and the growth generated while the asset was excluded from growth portfolios. There is no evidence that the growth generated while an asset is included in growth portfolios is systematically greater than the growth generated while the asset is excluded from growth portfolios. It can be concluded that the superior performance of growth portfolios comes not from their capacity to include assets during periods of high growth, but rather from their propensity to generally include high-growth assets and to exclude low-growth assets.

**Figure 7: Distribution of the Number of Assets Held in Growth Portfolios**

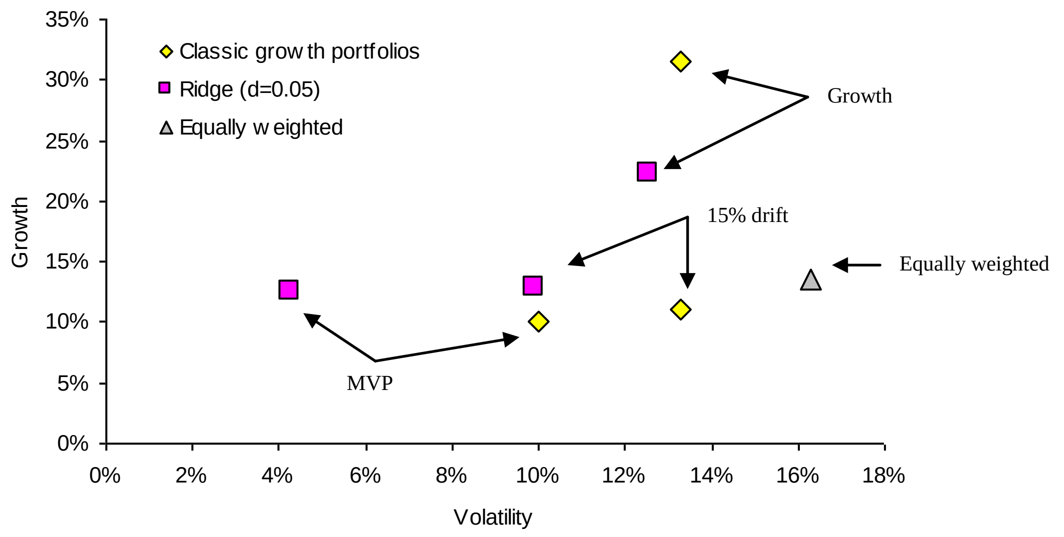



Increased transaction costs are a practical consideration for any strategy that results in the distribution of portfolio funds amongst a small number of assets. Transaction costs will be significant if a strategy requires the flip-flopping of large asset weights from one asset to another. One can see that this is the case to a degree with the no short-sales allowed classic growth portfolios. Table 6 shows that the maintenance of this growth optimal portfolio strategy would have required, on average, a turnover of stock of about 13% per month. The transaction costs associated with any strategy that turns over 13% of a portfolio per month are considerable, and will significantly lower the effective rate of portfolio growth. For example, the cost of portfolio adjustment would be 3.1%pa if a round transaction cost was 2% of traded value.

The ridge estimator was employed to produce short-sales allowed portfolio weights that were numerically stable and produced acceptable gearing levels. The justification for this use of a ridge estimator does not have the same force for short-sales not allowed portfolios. The no short-sales restriction considerably reduces the dimension of the covariance matrix that is inverted to produce portfolio weights. Classic short-sales not allowed portfolio weights are numerically stable and are by definition not geared. There is, however, an argument for the use of a ridge growth estimator in the short-sales not allowed context, on the grounds that it produces more diversified, less risky portfolios.

We have computed no short-sales, ridge, growth optimal portfolios using a ridge constant of 0.05 to facilitate comparison with the short-sales allowed results. Predictably, Table 6 reveals that the no short-sales allowed, ridge, growth portfolio performance lies somewhere between the performance of the classic growth optimal portfolio and that of the equally weighted portfolio. The no short-sales allowed, ridge, growth portfolio contains more assets (see Figure 7), is less risky and has a lower growth rate than the classic growth portfolio.

**Figure 8: Classic and Ridge Portfolio Performance**




## *Conclusion*

Growth optimal portfolio investment strategies were applied to a 25-year data set of **30** US companies. Initial statistical investigation of data provided no reason to be optimistic about the successful application of the growth techniques. The growth optimal technique assumptions of normality and stability were violated by the nature of the US data. Returns on the 30 stocks were found to be skewed and leptokurtic and to have time-varying variances and covariances. However, the growth optimal techniques performed well, despite the assumptions not being met.

The growth optimal portfolios, both short-sales allowed and short-sales not allowed, produced rates of growth that exceeded those of the benchmark portfolios. The classic no short-sales allowed, growth optimal portfolios produced impressive rates of growth that were more than double those of the benchmark portfolios. Analysis of the structure of these portfolios showed that, at any point in time, they consisted of a very small number of included stocks. The secret of the success of these portfolios appears to lie in their ability to select a few stocks during their high growth periods.

This study details the successful inclusion of a variant of ridge regression as the basis of a growth optimal strategy. The ridge growth optimal technique facilitated production of numerically stable weights for short-sales allowed portfolios. When short-sales were not allowed, the use of the ridge estimator produced more diversified growth portfolios.

There are two possible answers to the question of why the growth optimal techniques performed well in the face of non-normality and instability in the data. The first reason, which cannot be dismissed, is that the techniques worked well on this particular data set by pure chance alone. The second explanation is that the assumptions of normality and stability are not necessary to the success of the technique. While the model used in this paper assumes normality of the Ito process, it may be that growth investment strategy is equally efficacious under alternative stochastic processes that allow kurtosis. Why does the investment strategy cope with distributional instability? Perhaps the use of a moving window estimation process may counter the problems arising from the instability of mean growth rates and growth rate covariances.

This study details the successful application of growth optimal techniques. There is, however, no evidence of the general superiority of growth optimal techniques. Growth portfolio strategies are also high volatility strategies. While the results of an empirical study such as this are necessarily limited to the specific market and to the specific time-frame of the study, the point that this investigation makes is, however, that regardless of their other properties and potential drawbacks, the portfolios designed for maximal growth did in fact produce quite remarkable rates of growth.

## References

Berenson, M.L. and L.M. Levine, (1993), *Basic Business Statistics Concepts and Applications*, Prentice Hall.

Breiman, L. (1961), "Optimal Gambling Systems for Favourable Games", *Proceedings of Fourth Berkeley Symposium on Mathematical Statistics and Probability*, University of California Press, Berkeley

Cross J.E.; Barron A.R. (2003), "Efficient Universal Portfolios for Past–Dependent Target Classes, *Mathematical Finance,* April, vol. 13, 2, pp. 245-276

Cower, T.M. (1991), "Universal Portfolios", 1 (1), pp1-29

Cower, T.M. and J.A.Thomas (1996), "Universal Portfolios with Side Information", *IEEE Transactions on Information Theory*. 42(2), 348-363

Fernholz, R., Garvy R. and J.Hannon, (1998) "Diversity-Weighted Indexing", *Journal of Portfolio Management*, Winter, pp 74-82.

Golberger, A.S. (1964), *Econometric Theory*, Wiley.

Green, W.H. (1993), *Econometric Analysis (2nd Ed.)* McMillan.

Hakansson, N.H. (1971) "Multiperiod Mean-variance Analysis: Towards a General Theory of Portfolio Choice", *Journal of Finance*, 26, pp 857-884

Hull, J.C. (2000), *Options Futures and Other Derivatives* (4th Edition), Prentice Hall.

Hunt, B.F. (2002) "Growth Optimal Portfolios: their structure and nature", UTS mimeo.

Kelly, J.L. (1956) "A New Interpretation of the Information Rate", *Bell Systems Technical Journal*, 35, pp 917-926.

Levy, M., Levy, H. and Solomon, S. (2000), *Microscopic Simulation of Financial Markets*, Academic Press.

Luenberger, D.G. (1998), *Investment Science*, Oxford University Press, New York, New York.

Merton, R.C. and Samuelson, P.A. (1974), “Fallacy of the Log-normal Approximation to Optimal Portfolio Decision-making Over Many Periods”, *Journal of Financial Economics* 1, pp 67-94.

Pearson, E.S. (1969), “Some Comments on the accuracy of Box’s approximation to the distribution on M”, *Biometrica*, 56, pp 219-220.

Rachev, R. and Mittnik, S. (2000), *Stable Paretian Models in Finance*, Wiley.

# Endnotes

[1] Dividend adjusted prices were downloaded from the website http://chart.yahoo.com/.

[2] Most portfolio analysts would label the growth variable, g, as a continuously compounded return. This difference in terminology opens the door to possible confusion. Equations (1) and (2) show that the expected return over a very short period of time is $\mu \, \Delta t$. However, over a longer period of time the expected return is $(\mu - \sigma^2/2)\, t$. As Hull (2000), pp240-241 notes, "the term *expected return* is ambiguous. It can refer to either $\mu$ or $\mu - \sigma^2/2$". We will try to avoid this confusion by using the term *drift* to refer to short-period return, $\mu$, and using the term *growth* to refer to long-period return, $\mu$ -

[3] The skewness and kurtosis tests are based on the following. For a normally distributed random variable, x, the skewness coefficient, $\theta_1 = E[(x-\mu)^3]/\sigma^3$ estimated from a sample of size n, is distributed as $\theta_1 \approx N(0, 6/n)$. The coefficient of kurtosis, $\theta_2 = E[(x-\mu)^4]/\sigma^4$ is distributed as $\theta_2 \approx N(3, 24/n)$, where E is the expectation operator, $\mu$ is the mean and $\sigma$ is the standard deviation. The Jacque-Berra statistic, J, where:

$$J = n\left(\frac{\hat{\theta}_1^{\,2}}{6} + \frac{\left(\hat{\theta}_2 - 3\right)^3}{24}\right) \approx X^2(2)$$

[4] See Pearson (1969) p 219. An example of the use of M to test equality of covariance matrices can be found in Morrison (1976) pp 252-253 (note, however, the error in Morrison's equation (2)).

[5] Judge (1985) provides an exhaustive review of ridge estimators.