Investing Using Growth Optimal Portfolios

A Brief Explanation of the Technique and the a Dissemination of the Results of the Application of the Technique to Australian Top 100 Stocks

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Introduction

Any strategy that maximises the rate of growth of the value of an investment has an obvious and intuitive appeal to both the naïve and the professional investor. In an early application, Kelly [1956] proposed maximising the expected exponential growth rate of value of investment capital as an investment strategy in a gambling setting.

The so called Kelly system suggested gamblers allocate their wealth between a risk free asset, cash, and a risky, but favourable, gambling opportunity in a way that maximised the expected growth of capital. It has been shown [Breiman 1961] that the Kelly betting system is asymptotically optimal in that it minimises expected time to achieve any fixed value of terminal wealth and that it maximises rate of increase of wealth. Much that is proposed in the Kelly gambling system has direct application in a more traditional investment environment.

Hakkansson (1971) and Luenburger(1991) have justified the use of growth optimal portfolios on the basis of investor expected utility maximisation. It is comforting to know that there is a sound theoretical basis for advocating a growth portfolio investment strategy. However, the Kelly view, that maximising investment growth of value is a self-evident superior strategy, probably resonates more with the investment sector.

The application of the Kelly system to an n-asset investment portfolio environment, where a risk-free asset may or may not exist and where returns are normally distributed, is straightforward. Investors adhering to the Kelly method choose asset weights, **w**, that maximise portfolio expected growth and by so doing construct portfolios that at once:

1. maximise expected terminal value $S_p(T)$ for any time T ,
2. minimise the expected time required for terminal value to reach any specified threshold value,.
3. are always more likely to have a value in excess of any other portfolio at any point of time during the investment period.

Technical Stuff

What are *Growth Optimal* portfolios and how are they constructed? Portfolio expected growth, g_p, is defined as

$$g_p = r_p - \frac{1}{2}\sigma_p \qquad (1)$$

Where, r_p is the portfolio expected return and σ_p^2 is the variance of portfolio return. It is evident from equation (1) that portfolio growth is higher the;

- higher is portfolio return and the
- lower is portfolio variance

The connection to higher return and lower variance is a an indication that *Growth Optimal* portfolios are indeed Makowitz efficient portfolios.

Equation (1) can be expanded to express portfolio expected growth in terms of individual asset returns, variances and covariace;

$$g_p = \left(r_p - \frac{1}{2}\sigma_p^2 \right) = \left(\mathbf{w}^T\mathbf{r} - \frac{1}{2}\mathbf{w}^T\boldsymbol{\Omega}\,\mathbf{w} \right) \qquad (2)$$

Where, w is a vector of individual asset weights, **Ω** is an nxn matrix of variances and covariances and **r** is a vector of individual asset expected returns.

$$w = \begin{pmatrix} w_1 \\ .. \\ .. \\ w_n \end{pmatrix}, \quad r = \begin{pmatrix} r_1 \\ .. \\ .. \\ r_n \end{pmatrix}, \qquad \Omega = \begin{pmatrix} \sigma_{1,1} & . & \sigma_{1,n} \\ . & . & . \\ \sigma_{n,1} & . & \sigma_{n,n} \end{pmatrix}$$

Growth Optimal portfolios are constructed by choosing weights, **w,** that maximise g_p. subject to the no short-sales constraint that $w_i \geq 0$. An algebraic solution to the growth maximum problem posed in (2) does not exist where short-sales are not permitted. In practice, weights that maximise portfolio growth are determined using a quadratic programming technique.

The *Growth* portfolios, true to their design intention, have the following properties.

1. *Growth* portfolios are expected to have a higher value than any other alternative portfolios at any time in the future.
2. *Growth* portfolio minimises the expected time taken for a portfolio to reach any given terminal value.

Growth Optimal Portfolio Characteristics

Growth optimal portfolios are high return, high risk Markowitz efficient portfolios. Typically they contain only a few assets. Advantages of investment funds composed of *Growth Optimal* portfolios include;

- Rapid growth of fund value
- Low correlation with other assets or funds
- Low management costs

The obvious detraction of Growth Optimal Portfolios is the;

- High volatility of fund return

The high volatility of *Growth* portfolios is a a negative attribute. There is however a technique available for reducing volatility (at the expense of lower average returns) by increasing the number of stocks held within the Growth portfolios. This technique simply adds a factor to the diagonal elements of the asset returns covariance matrix. Here a modified covariance matrix Ω* replaces Ω in the *Growth Optimal* portfolio technique.

$$\Omega^* = \begin{pmatrix} \sigma_{1,1} + d & . & \sigma_{1,n} \\ . & . & . \\ \sigma_{n,1} & . & \sigma_{n,n} + d \end{pmatrix}$$

Application to ASX S&P 100 stocks

The following results have been obtained by back testing the *Growth* technique on 10 years of daily Australian price data on ASX 100 stocks[1] The figures below are the results of back testing varying two input variable:

Estimation period. The expected returns and covariance matrix have to be estimated from prior data. The estimation period is varied from 6 month to one year

Diagonal diversity factor. This is the factor added to the diagonal elements of the covariance matrix to increase the number of stocks in the Growth portfolios.

The back testing exercise was conducted as follows

1. at the end of the focus month, past data was employed to estimate average returns and the covariance matrix over the previous x months.
2. these estimates were used to calculate the assets and weights comprising the *Growth* portfolio
3. The return on the *Growth* portfolio was computed over the next month and
4. then the time period was advanced a month and steps 1 – 3 were repeated.

Estimation period	Diag Factor	Median No. Stocks	Max No. Stocks	Average (%pa)	Volatility (%pa)	Worst 3 months (%)	Worst 6 months (%)	Transaction costs (%pa)
6-month		1	3	35.5%	30.8%	-35.70%	-31.04%	1.71%
9-month		1	4	38.4%	29.0%	-23.52%	-15.77%	1.33%
12-month		1	4	41.3%	28.8%	-28.30%	-25.85%	1.11%
12-month	33%	4	8	34.3%	22.1%	-16.87%	-16.14%	0.99%
12-month	66%	6	14	32.5%	19.8%	-15.47%	-14.09%	0.89%

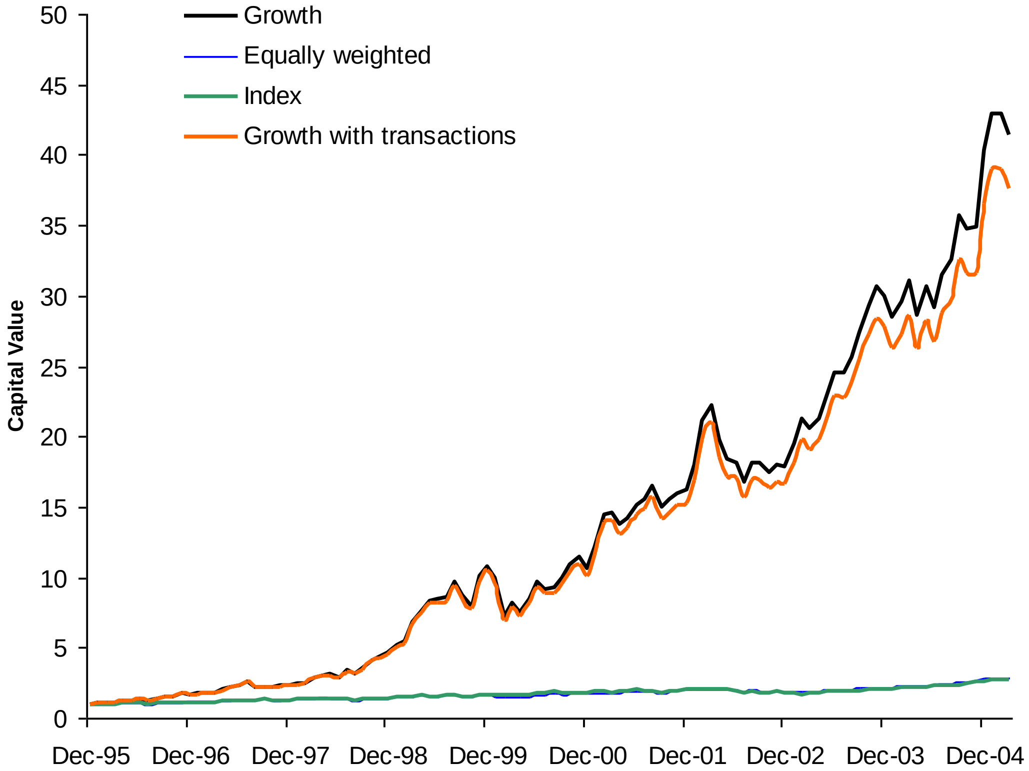
The performance of the *Growth* portfolios can be judged against the two bench mark portfolios of

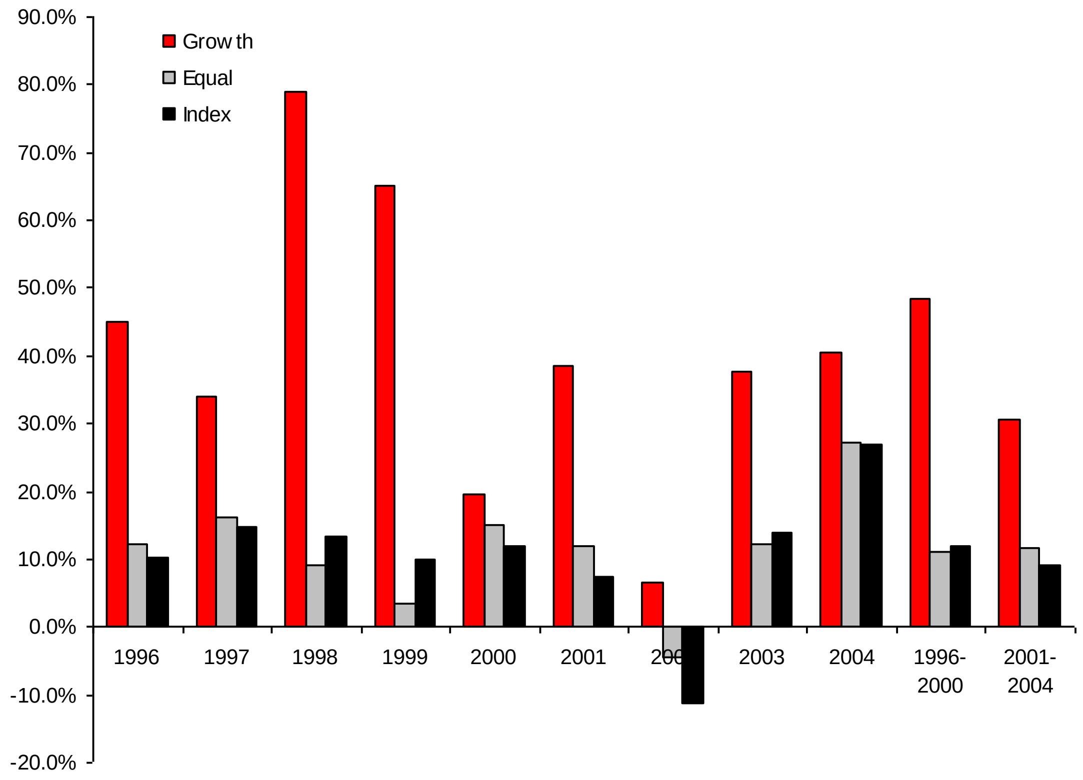
1. equally weighted portfolio
2. the ASX 100 index

[1] The stocks included in the back testing are those that were included in the ASX 100 index as at June 2004. Of course a number of these stocks did not exist in 1995. In fact only 63 stocks are included as possible investments in 1995.

	Average (%pa)	Volatility (%pa)	Worst 3 months (%)	Worst 6 months (%)
Equally	11.9%	10.2%	-9.12%	-8.41%
Index	10.9%	12.1%	-11.85%	-11.64%

The superior performance of the Growth portfolios is exampled by the record of the 12-month, no diagonal factor *Growth* portfolio.





It is interesting note the relationship the various Growth portfolios for the most recent data at the end of March 2005.

***Growth* Portfolio at 31st March 2005**		
Estimation period	Diag Factor	Portfolio
6-month		OSH (86%), COH (14%)
9-month		ALL (100%)
12-month		ALL (100%)
12-month	33%	ALL (87%), OSH (13%)
12-month	66%	ALL (66%), OSH (24%), COH (14%)

The *Growth* portfolio technique is no method for turning lead into gold. It is simply a high return, high risk strategy. This said, the results of the application of the method to ASX data suggest that this technique may be suitable for some investors.