Growth Optimal Portfolios: their structure and nature

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Any investment strategy that maximises portfolio growth has an intuitive appeal for both the professional and non-professional investor. This paper investigates the structure and properties of growth oriented portfolios using a matrix algebra approach. Data on the returns of five Australian listed companies are used in the paper to provide specific illustrations of growth portfolio properties.

This paper highlights the nexus between growth optimal portfolios and Markowitz mean-variance portfolios. Growth portfolios are shown to be Markowitz efficient portfolios. The paper contrasts the properties of maximum growth portfolios with those of minimum variance portfolios.

Finally, following Long (1990) we derive the growth optimal portfolio equivalent of the capital asset pricing model with the growth optimal portfolio acting as a pricing numeraire.

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Introduction

Any strategy that maximises the rate of growth of the value of an investment has an obvious and intuitive appeal to both the naïve and the professional investor. In an early application, Kelly [1956] proposed maximising the expected exponential growth rate of value of investment capital as an investment strategy in a gambling setting.

The so called Kelly system suggested gamblers allocate their wealth between a risk free asset, cash, and a risky, but favourable, gambling opportunity in a way that maximised the expected growth of capital. It has been shown [Breiman 1961] that the Kelly betting system is asymptotically optimal in that it minimises expected time to achieve any fixed value of terminal wealth and that it maximises rate of increase of wealth. Much that is proposed in the Kelly gambling system has direct application in a more traditional investment environment.

Let us examine the growth characteristics of more traditional investments. It is well known that if the passage of an asset price, S, through time, t, is governed by geometric Brownian motion (generalised Weiner process) [1]

$$dS(t) = r\,S(t)\,dt + \sigma S(t)\,dz \tag{1}$$

where r is the rate of drift and z is a standard Weiner process, then the expected growth of the asset, G, over time t, can be expressed as:

$$G = E\left[\ln\left(\frac{S(t)}{S(0)}\right)\right] = \left(r - \frac{1}{2}\sigma^2\right)t \tag{2}$$

The combinatorial properties of normal random variables dictate that if the value of n assets follow a geometric Brownian motion, the value of a combination of these assets, S_p, defined by a portfolio weights vector, $\mathbf{w}^T = (w_1, \ldots, w_n)$ will also be characterised by geometric Brownian motion[2]

$$dS_p(t) = r_p S_p(t)\,dt + \sigma_p\, S_p(t)\,dz \tag{3}$$

and will have an expected portfolio growth rate per unit of time, g_p, where:

[1] For example see Luenberger D.G. (1998) pp 310-313.

[2] It should be acknowledged that there is a wealth of evidence to suggest equity returns, while they are independent, are not normally distributed. In particular, many equity returns exhibit strong kurtosis. (See Rachev and Mittnik (2000), pp 605-616 for a summary of alternative distributional models.) It ought to be noted that most of the results presented in this paper do not rely on normality but rather only require that asset returns have finite variances.

$$
\begin{aligned}
g_p &= \frac{G_P}{t} \\
&= \left(r_p - \frac{1}{2}\sigma_p^2 \right) \\
&= \left(\mathbf{w}^T\mathbf{r} - \frac{1}{2}\mathbf{w}^T\boldsymbol{\Omega}\,\mathbf{w} \right)
\end{aligned}
\tag{4}
$$

where $\boldsymbol{\Omega}$ is an nxn matrix of variances and covariances and $\mathbf{r}$ is a "returns" vector of individual expected drift rates per unit of time, $r^T=(r_1, \ldots, r_n)$.[3]

Hakkansson (1971) and Luenburger(1991) have justified the use of growth optimal portfolios on the basis of investor expected utility maximisation. It is comforting to know that there is a sound theoretical basis for advocating a growth portfolio investment strategy. However, the Kelly view, that maximising investment growth of value is a self-evident superior strategy, probably resonates more with the investment sector.

The application of the Kelly system to an n-asset investment portfolio environment, where a risk-free asset may or may not exist and where returns are normally distributed, is straightforward. Investors adhering to the Kelly method choose asset weights, **w**, that maximise portfolio expected growth and by so doing construct portfolios that at once:

1. maximise expected terminal value $S_p(T)$ for any time T ,
2. minimise the expected time required for terminal value to reach any specified threshold value,.
3. are always more likely to have a value in excess of any other portfolio at any point of time during the investment period.

The purpose of this paper is not to justify the use of growth focussed portfolios, but rather to explore the properties of growth portfolios. The Markowitz (1952) mean-variance approach to portfolio selection concentrates on portfolio expected return and variance of return. The contribution of this paper is to highlight to results relating to another important of portfolio characteristic namely, portfolio expected growth. In doing so we expose the nexus that exists between growth portfolios and Markowitz mean-variance efficient portfolios.

Growth – a Key Portfolio Characteristic

The expected growth of any portfolio is another important portfolio characteristic like the more familiar portfolio expected return and the variance of portfolio return. Assuming a Weiner asset process, portfolio expected growth is a simple function of these latter two portfolio characteristics being equal to the portfolio expected return less half portfolio variance.

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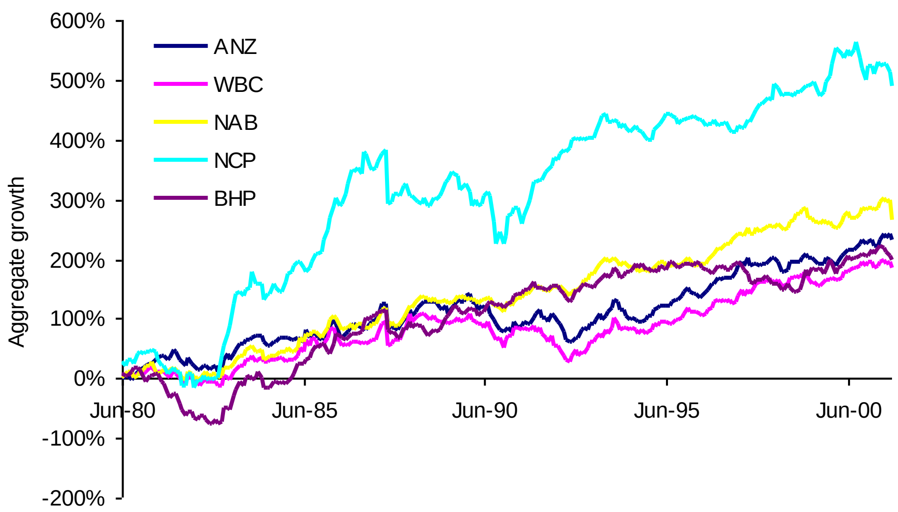
[3] The expected return over a very short period of time is $\mu\,\Delta t$. However, over a longer period of time the expected return is $\mu - \sigma^2/2$. As Hull (2000), pp240-241 notes, "the term *expected return* is ambiguous. It can either refer to μ or $\mu - \sigma^2/2$". When the term expected return (or symbol r) is used in this paper it is in reference to the drift term, μ, in a generalised Weiner process.

Price data on five Australian companies is employed to illustrate some of the properties of growth portfolios. The companies are Australian and New Zealand Banking Ltd. (ANZ), Westpac Banking Corporation (WBC), National Australia Bank (NAB), News Corporation (NCP) and BHP Billiton (BHP). These companies are the five largest (by market capitalisation) Australian companies that continually traded over the data period of June 1980 to September 2001. Figure 1 records the growth of the five stocks over the two decade period. Further statistics relating to the five companies is set out in the appendix.

"Efficient" growth portfolios may be derived in the same way that efficient variance portfolios are derived within a Markowitz framework. [4] In a Markowitz framework portfolio variance is minimised subject to a constraint that portfolio expected return is equal to an arbitrary constant.

In a growth framework, portfolio expected growth is maximised subject to a constraint that portfolio expected return is equal to an arbitrary constant. That is, the expected portfolio growth rate, $g_p = \mu_p - \sigma^2_p/2$, is maximised subject to μ_p equalling some constant, k. The dual of this problem is the minimisation of portfolio variance, σ^2_p, subject to portfolio return being set to the same arbitrary constant, k. Examination of the dual of the growth problem reveals that efficient growth portfolios are also Markowitz, mean variance efficient portfolios with individual growth constants, $\mu_{I,}$ substituting for Markowitz average return.

Figure 1: Aggregate Growth

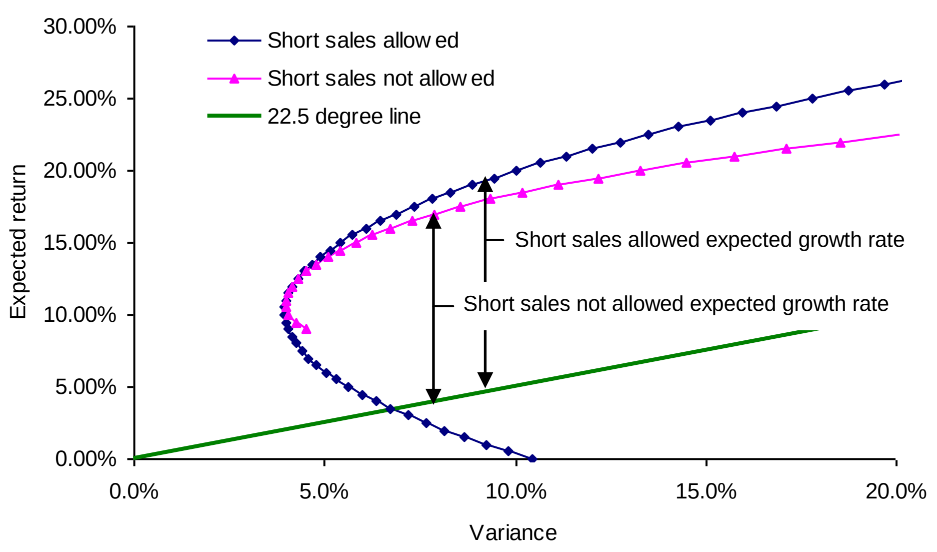


The point was made above that growth portfolio analysis is firmly set within the portfolio mean variance framework. This is illustrated in Figure 2. Figure 2 contains three plots. The two quadratic "minimum variance" lines represent the expected return-variance trade-off for both short-sales allowed and short-sales not allowed,

[4] In the analysis follows, portfolios that focus on growth while controlling other factors are henceforth termed *growth efficient portfolios*, and portfolios that have the absolute maximum portfolio expected growth, regardless of other factors, are termed *growth optimal portfolios* (GOP).

minimum variance portfolios consisting of the five Australian equity assets listed in Table 1. The 22.5^{o} straight line in Figure 1 plots r =-σ^2/2.

Figure 2: Return and Risk



The vertical distance between the mean-variance portfolio return and the r = σ^2/2 line represents the expected growth of an efficient portfolio.

Short sales allowed, efficient growth portfolios are asset weight vectors that maximise portfolio growth subject to two restrictions (1) that the weights vector sums to unity and (2) that portfolio expected return is equal to some arbitrary constant, k. The two constraint equations have a matrix equation representation:

$$\begin{pmatrix} 1,\ldots\ldots 1 \\ r_1 \ldots\ldots r_n \end{pmatrix} \begin{pmatrix} w_1 \\ . \\ w_n \end{pmatrix} = \begin{pmatrix} 1 \\ k \end{pmatrix} \tag{5}$$

or

$$\mathbf{B}\,\mathbf{w} = \mathbf{b}$$

where B is a (2xn) matrix of 1s and expected returns, $r_{i\,,}$ i = 1 ....n , and b is a (2x1) vector of 1 and the desired minimum variance return, k. Similarly, selection of an efficient growth portfolio can be reduced to the identification of a weights vector, **w**, that maximises

$$g = \mathbf{w}^T\mathbf{r} - \tfrac{1}{2}\mathbf{w}^T\mathbf{\Omega}\mathbf{w} \tag{6}$$

subject to the restrictions in (5). Inspection of (5) and (6) reveals that the problem reduces to one of minimising portfolio variance, the second term on the rhs of (6), subject to the restrictions in (5).

Minimisation of $\sigma^2_p = \mathbf{w}^T \boldsymbol{\Omega}\, \mathbf{w}$, subject to $\mathbf{B\, w} = \mathbf{b}$, has the solution:[5]

$$\mathbf{w}_k = \boldsymbol{\Omega}^{-1}\mathbf{B}^T(\mathbf{B}\boldsymbol{\Omega}^{-1}\mathbf{B}^T)^{-1}\mathbf{b} \tag{7}$$

where w_k is the portfolio which, of all portfolios having an expected return of k, has the least possible portfolio variance. This variance is:

$$\begin{aligned}\sigma_k^2 &= \mathbf{w}_k^T\,\boldsymbol{\Omega}\,\mathbf{w}_k \\ &= \mathbf{b}^T(\mathbf{B}\,\boldsymbol{\Omega}^{-1}\mathbf{B}^T)^{-1}\mathbf{b}\end{aligned} \tag{8}$$

Simplifying (8) by defining a (2x2) matrix $\mathbf{A} = (\mathbf{B}\,\boldsymbol{\Omega}^{-1}\mathbf{B}^T)^{-1}$ and enumerating the contents of **b**, produces:

$$\begin{aligned}\sigma_k^2 &= (1,\ k)\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2}\end{pmatrix}\begin{pmatrix}1 \\ k\end{pmatrix} \\ &= a_{1,1} + 2a_{2,1}k + a_{2,2}k^2\end{aligned} \tag{9}$$

Equation (9) is a quadratic expression relating the variance of the minimum variance portfolio to the portfolio expected return. The expected growth of this portfolio can thus be expressed in terms of the arbitrary portfolio return constant, k as

$$\begin{aligned}g_k &= r_k - \tfrac{1}{2}\sigma_k^2 \\ &= -\tfrac{1}{2}a_{1,1} + (1 - a_{2,1})\,k - \tfrac{1}{2}a_{2,2}\,k^2\end{aligned} \tag{10}$$

Equation (10) shows that growth efficient portfolios are a quadratic function of portfolio return. The quadratic relationship between portfolio expected growth and portfolio return is evident in Figure 3, which shows the relationship for both short sales allowed and short sales not allowed portfolios constructed using the application data.

Another property of efficient growth portfolios is that expected growth is a quadratic function of expected return. The quadratic relationship between growth and return, evident in equation (10), is depicted for the Australian equities data in Figure 3.

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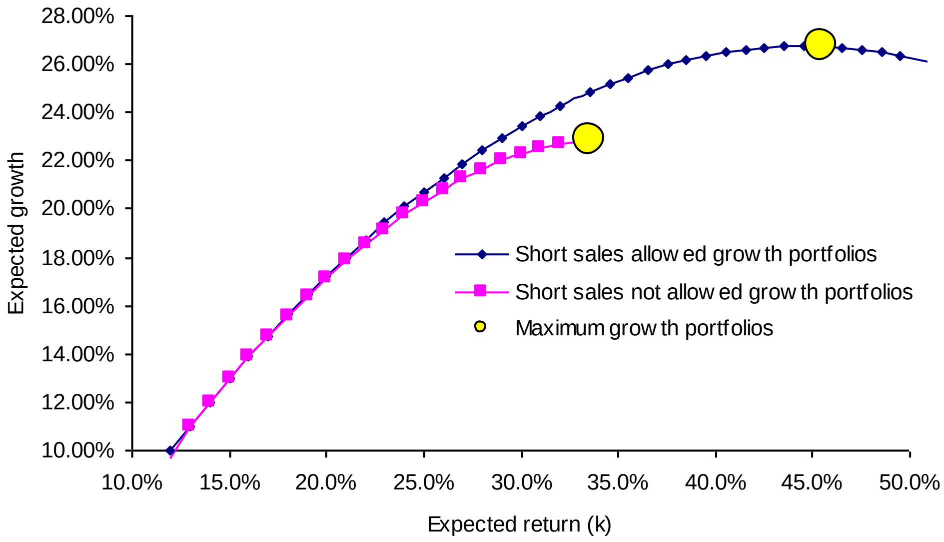
[5]Differentiating the Lagrange function, $L = w'\Omega w - \lambda(Bw - b)$ produces:

$$\begin{pmatrix}\delta L / \delta w \\ \delta\lambda / \delta w\end{pmatrix} = \begin{pmatrix}\Omega w - B'\lambda \\ Bw - b\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix}$$

solving the above equation by partitioned inversion produces a solution for w. (see Golberger, (1964).

Figure 3: Quadratic growth



Each of the portfolios that underlie the plots in Figure 3 is on a maximum growth frontier as they all have maximal growth given a particular portfolio return, k. The return associated with the absolute, maximum growth (growth optimal), short sales allowed, portfolio point can deduced by differentiating equation (10)

$$\frac{dg_k}{dk} = (1 - a_{1,2}) - a_{2,2}k \tag{11}$$

setting the derivative to zero produces the return, $k_{opt,}$ associated with the growth optimal portfolio:

$$k_{GOP} = \frac{1 - a_{1,2}}{a_{2,2}} = \frac{1}{a_{2,2}} + k_{MVP} \tag{12}$$

Where, k_{MVP} is the expected return attached to the minimum variance portfolio[6].

Using the application data, $\mathbf{A} = (\mathbf{B}\,\boldsymbol{\Omega}^{-1}\mathbf{B}^{T})^{-1}$ matrix is evaluated as:

$$\mathbf{A} = \begin{pmatrix} 0.0923 & -0.4089 \\ -0.4089 & 3.1649 \end{pmatrix}$$

Thus k_{GOP} = (1+0.4089)/3.1649 = 44.52% and can be contrasted with the Markowitz absolute minimum variance return, $k_{MVP} = -0.4089/3.1649 = 12.91\%$

It can be seen from Figure 3 that the application data yields a corner solution for the short sales not allowed maximal growth portfolio[7]. The short sales not allowed growth optimal portfolio consists of 100% of a single asset, NCP. Of course this is

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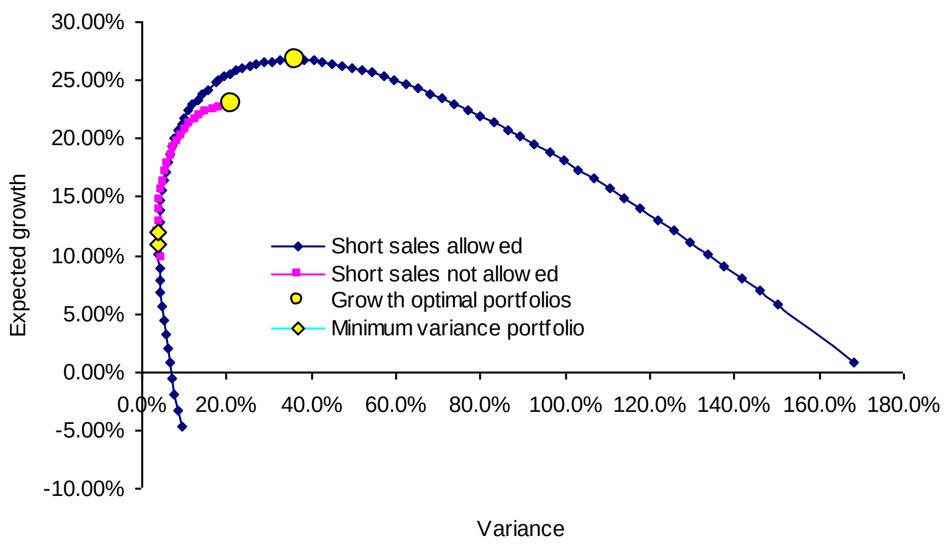
[6] Setting the first differential of equation (14) to zero produces $k_{MVP} = -a_{2,1}/a_{2,2}$.
[7] Empirical short sales not allowed portfolios were obtained using Microsoft Excel's solver module.

not a general result portfolio and is a direct result of the small number of assets in the application data set. However it does illustrate an important aspect of the construction of growth optimal, short sales not allowed, portfolios, namely that the risk reduction benefits of diversification are not necessarily sufficient reason to justify the inclusion of additional assets. Taking the highest yielding asset as a minimal, short sales not allowed, growth efficient portfolio starting point, the inclusion of an additional, lower yielding asset is only justified when the benefit of additional diversification exceeds the reduction in overall portfolio yield that accompanies the inclusion of the lower yielding asset.

A growth-variance frontier can be drawn in risk space like the more familiar Markowitz mean-variance frontier. A growth-variance frontier for the Australian equity data is set out in Figure 4. Figure 4 exhibits a "surfboard skeg" shape typical of growth-variance frontier plots.

Figure 4: Growth, risk curve



We can gain some insight into the individual asset structure of growth portfolios by defining a (nx2) vector, $\mathbf{D} = \mathbf{\Omega}^{-1}\mathbf{B}^T(\mathbf{B}\mathbf{\Omega}^{-1}\mathbf{B}^T)^{-1}$ and by substituting this term in equation (11). This enables us to express individual asset weights as functions of k:

$$
\begin{aligned}
\mathbf{w}_k &= \mathbf{\Omega}^{-1}\mathbf{B}^T(\mathbf{B}\mathbf{\Omega}^{-1}\mathbf{B}^T)^{-1}\mathbf{b} \\
\mathbf{w}_k &= \mathbf{D}\mathbf{b} \\
\begin{pmatrix} w_1 \\ . \\ w_n \end{pmatrix} &= \begin{pmatrix} d_{1,1} & d_{1,2} \\ & \\ d_{n,1} & d_{n,2} \end{pmatrix} \begin{pmatrix} 1 \\ k \end{pmatrix} \\
\mathbf{w}_k &= \mathbf{d}_{.1} + k\,\mathbf{d}_{.2}
\end{aligned} \tag{13}
$$

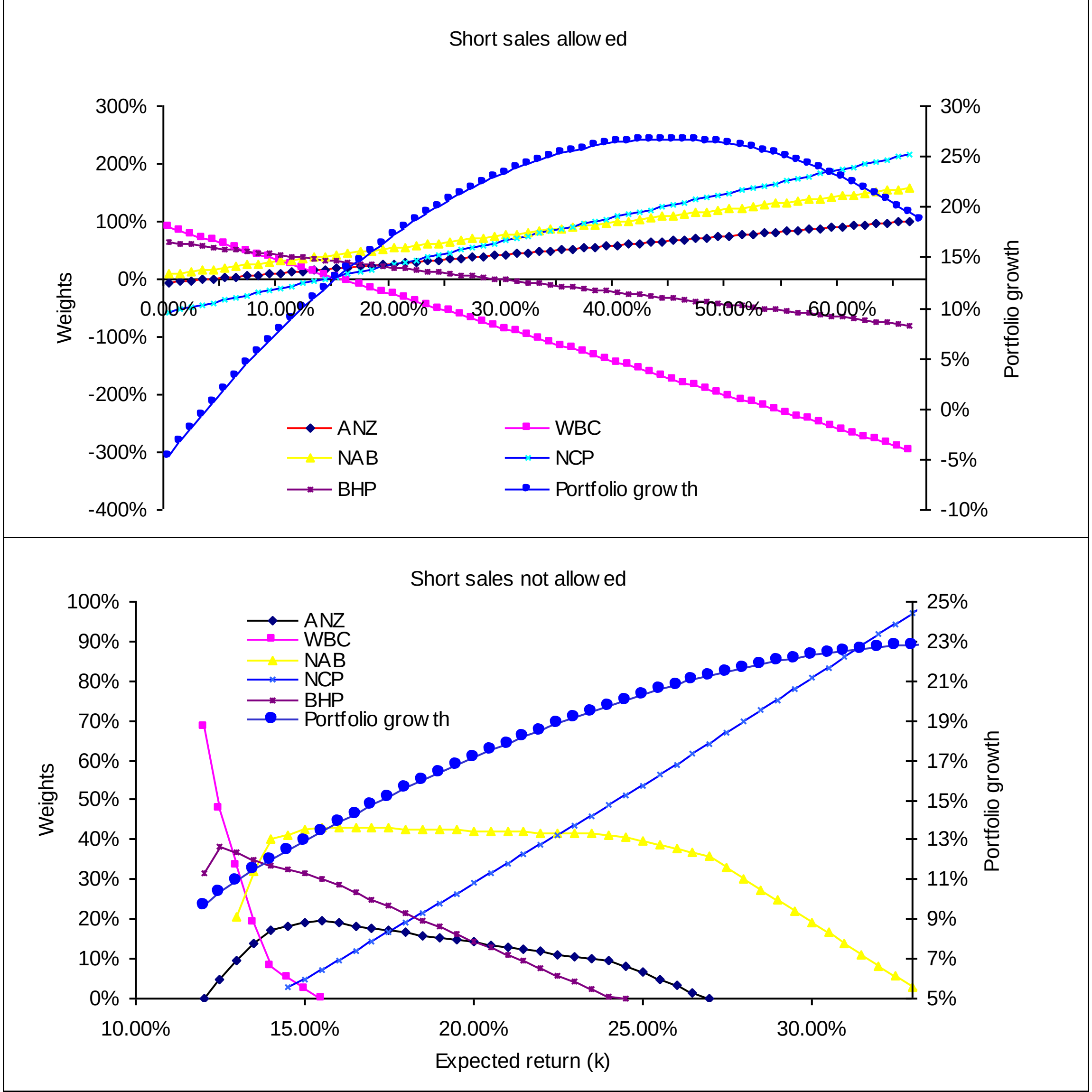
where $d_{.i}$ is the i[th] column of **D**.

Rewriting equation (13) shows that the weight of an individual asset in an efficient growth portfolio $\mathbf{w}_k$ is a linear function of k.

$$\mathbf{w}_i = d_{i,1} + k\, d_{i,2} \tag{14}$$

Thus the weight of an individual asset in a minimum variance portfolio, is either an increasing or decreasing linear function of portfolio return. This phenomenon is illustrated in Figure 5 for both short sales allowed and short sales not allowed growth portfolios.

Figure 5: Growth portfolio weights



Note that the short sales not allowed growth efficient portfolios are simply a series of short sales allowed growth efficient portfolios with various numbers of included assets. In other words, the set of short sales not allowed growth portfolios consists of a number of sub-sets of short sales allowed growth portfolios.

Growth portfolios, like mean-variance portfolios obey a "two fund" rule. That is, all growth efficient portfolios may be generated from a linear combination of two other efficient growth portfolios. Consider two efficient growth portfolios having arbitrary expected returns k_1 and k_2. A third growth portfolio constructed by combing a portion, c, from the first portfolio and (1-c) from the second portfolio is also a growth efficient portfolio.

$$\begin{aligned}\mathbf{w}_{k_3=c\,k_1+(1-c)\,k_2} &= c\,\mathbf{w}_{k_1}+(1-c)\mathbf{w}_{k_2}\\ &= \mathbf{d}_{.,1}+(c\,k_1+(1-c)k_2)\,\mathbf{d}_{.,2}\end{aligned} \tag{15}$$

Growth Optimal Portfolio Structure

Selecting a weight vector, **w**, that maximises the expected portfolio growth, $g=\mathbf{w}^{\mathrm{T}}\mathbf{r}-\frac{1}{2}\mathbf{w}^{\mathrm{T}}\boldsymbol{\Omega}\mathbf{w}$, subject to unity sum constraint on the weights, identifies the structure of the growth optimal portfolio. Differentiating the Lagrangrean expression:

$$L=\mathbf{w}^{\mathrm{T}}\mathbf{r}-\tfrac{1}{2}\mathbf{w}^{\mathrm{T}}\boldsymbol{\Omega}\mathbf{w}-\lambda(\boldsymbol{\iota}^{\mathbf{T}}\mathbf{w}-1)$$

produces:

$$\frac{\delta L}{\delta \mathbf{w}}^{\mathrm{T}}=\mathbf{r}-\boldsymbol{\Omega}\mathbf{w} \text{ and } \frac{\delta L}{\delta\lambda}=\boldsymbol{\iota}^{\mathbf{T}}\mathbf{w}-1 \tag{16}$$

where $\boldsymbol{\iota}$ is a vector of units. Setting the equations in (16) to zero to satisfy the first order conditions and rearranging the equations produces:

$$\begin{pmatrix}\boldsymbol{\Omega} & \boldsymbol{\iota}\\ \boldsymbol{\iota}^{\mathrm{T}} & 0\end{pmatrix}\begin{pmatrix}\mathbf{w}\\ \lambda\end{pmatrix}=\begin{pmatrix}\mathbf{r}\\ 1\end{pmatrix} \tag{17}$$

The solution to (17) can be expressed as:

$$\begin{pmatrix}\mathbf{w}\\ \lambda\end{pmatrix}=\begin{pmatrix}\mathbf{E}^{1,1} & \mathbf{E}^{1,2}\\ \mathbf{E}^{2,1} & \mathbf{E}^{2,2}\end{pmatrix}\begin{pmatrix}\mathbf{r}\\ 1\end{pmatrix} \tag{18}$$

where the $E^{i,j}$ are the elements of the inverse of the left hand matrix in (17). Thus, the GOP weights, **w***, vector has the following structure:

$$\mathbf{w^*}=\mathbf{E}^{1,1}\,\mathbf{r}+\mathbf{E}^{1,2} \tag{19}$$

where:

$$\mathbf{E}^{1,1}=\boldsymbol{\Omega}^{-1}\left(\mathbf{I}-\frac{\boldsymbol{\iota}\,\boldsymbol{\iota}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}}{\boldsymbol{\iota}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\boldsymbol{\iota}}\right) \tag{20}$$

and

$$\mathbf{E}^{1,2}=\frac{\boldsymbol{\Omega}^{-1}\boldsymbol{\iota}}{\boldsymbol{\iota}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\boldsymbol{\iota}} \tag{21}$$

Naturally there is some predictable structure evident in the growth optimal portfolio equation (19).

It is easily shown that that sum of the elements of each column of $\mathbf{E}^{1,1}$ is zero and that the sum of the $\mathbf{E}^{1,2}$ vector equals unity. These two results guarantee that the weights vector, **w**, sums to unity as required. The short sales allowed and the short sales not allowed absolute maximum growth portfolio weights ($w*_{ssa,}$ and , $w*_{ssns,}$ respectively) for the application data are:

$$\mathbf{w*}_{ssa} = \begin{pmatrix} w*_{ANZ} \\ w*_{WBC} \\ w*_{NAB} \\ w*_{NCP} \\ w*_{BHP} \end{pmatrix} = \begin{pmatrix} 0.66 \\ -1.71 \\ 1.09 \\ 1.27 \\ -0.33 \end{pmatrix} \text{ and } \mathbf{w*}_{ssna} = \begin{pmatrix} 0.00 \\ 0.00 \\ 0.00 \\ 1.00 \\ 0.00 \end{pmatrix}$$

$\mathbf{E}^{1,1}$ is a symmetric, positive semi-definite matrix.[8] Thus the diagonal elements of $\mathbf{E}^{1,1}$ are positive and the off-diagonal elements in any column or row of $\mathbf{E}^{1,1}$ are net negative. These properties are evident in the empirical, $\mathbf{E}^{1,1}$ of the example:

$$\mathbf{E}^{\mathbf{1,1}} = \begin{pmatrix} 37.86 & -21.42 & -11.42 & 0.48 & -5.50 \\ -21.42 & 49.27 & -20.56 & -2.61 & -4.67 \\ -11.42 & -20.56 & 38.07 & -1.55 & -4.54 \\ 0.48 & -2.61 & -1.55 & 6.31 & -2.63 \\ -5.50 & -4.67 & -4.54 & -2.63 & 17.34 \end{pmatrix}$$

The positive diagonal elements of $\mathbf{E}^{1,1}$ guarantee that any increase in the growth rate of any individual asset r_i results in an increase in the weighting of that asset in the GOP, as it can be seen from (19) that:

$$\frac{\delta\mathbf{w}}{\delta\mathbf{r}} = \mathbf{E}^{1,1} \tag{22}$$

Growth Optimum Portfolio Characteristics

Any portfolio has three important characteristics. These are the portfolio expected return, portfolio variance and portfolio expected growth. The expected return associated with a growth optimal portfolio can be expressed as:

$$\mathbf{r*} = r^T\mathbf{w*} = r^T(\mathbf{E}^{1,1}\,\mathbf{r} + \mathbf{E}^{1,2}) \tag{23}$$

The variance of a growth optimal portfolio has a relatively simple expression. Recognising that $\mathbf{E}^{1,1}\Omega\mathbf{E}^{1,2} = \mathbf{0}$, $\mathbf{E}^{1,1}\Omega\mathbf{E}^{1,1} = \mathbf{E}^{1,1}$ and $\mathbf{E}^{1,2}\Omega\mathbf{E}^{1,2} = 1/\,d$, where d is the sum of elements of the inverse of the covariance matrix, ie $d = \boldsymbol{\iota}^T\boldsymbol{\Omega}^{-1}\boldsymbol{\iota}$

[8] See Appendix

$$
\begin{aligned}
\sigma^{*2} &= \mathbf{w}^{*\mathrm{T}}\,\Omega\,\mathbf{w}^{*} \\
&= (\mathbf{E}^{1,1}\mathbf{r}+\mathbf{E}^{1,2})^{\mathrm{T}}\Omega(\mathbf{E}^{1,1}\mathbf{r}+\mathbf{E}^{1,2}) \qquad (24)\\
&= \mathbf{r}^{\mathrm{T}}\mathbf{E}^{1,1}\mathbf{r}+\tfrac{1}{d}
\end{aligned}
$$

The GOP portfolio rate of growth:

$$
\begin{aligned}
g &= r^{*}-\tfrac{1}{2}\sigma^{*} \\
&= \mathbf{r}^{\mathrm{T}}(\tfrac{1}{2}\mathbf{E}^{1,1}+\mathbf{E}^{1,2})-\tfrac{1}{d}
\end{aligned}
\qquad (25)
$$

The portfolio characteristics for the application data are set out in Table 2.

Table 2: GOP portfolio characteristics

	Short sales allowed	Short sales not allowed
Expected return	44.52%	33.53%
Variance	35.55%	21.30%
Volatility	59.62%	46.15%
Growth	26.74%	22.87%

Comparison with MVP portfolio

The absolute minimum variance portfolio (MVP) results from the removal of the expected return equation from the restriction equation (5). The removal of the expected return restriction reduces the restriction equation **Bw**=**b** to $\boldsymbol{\iota}^{\mathrm{T}}\mathbf{w}=1$. Substituting the reduced restriction equation into equation (7) produces the vector of absolute minimum variance portfolio weights, w^{o}:

$$
w^{o} = \frac{\boldsymbol{\Omega}^{-1}\boldsymbol{\iota}}{\boldsymbol{\iota}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\boldsymbol{\iota}} = \mathbf{E}^{1,2} \qquad (26)
$$

Some simple algebra, using the structure of w^{o}, produces formulae for MVP expected return, r^{o}, MVP variance, σ^{2}_{o}, and MVP expected growth, g^{o}. The formulae for these MVP portfolio characteristics are set out in Table 3.

Table 3: A comparison of MVP and GOP portfolio characteristics

	MVP portfolio	GOP portfolio
Weights	$\mathbf{w}^{o} = \mathbf{E}^{1,2}$	$\mathbf{w}^{*} = \mathbf{E}^{1,1}\mathbf{r}+\mathbf{E}^{1,2}$
Expected return	$r^{o} = \mathbf{r}^{\mathrm{T}}\mathbf{E}^{1,2}$	$r^{*} = \mathbf{r}^{\mathrm{T}}(\mathbf{E}^{1,1}\mathbf{r}+\mathbf{E}^{1,2})$
Variance	$\sigma_{0}^{2} = \frac{1}{d}$	$\sigma_{*}^{2} = \mathbf{r}^{\mathrm{T}}\mathbf{E}^{1,1}\mathbf{r}+\frac{1}{d}$
Growth	$g^{o} = \mathbf{r}^{\mathrm{T}}\mathbf{E}^{1,2}-\frac{1}{d}$	$g^{*} = \mathbf{r}^{\mathrm{T}}(\frac{1}{2}\mathbf{E}^{1,1}+\mathbf{E}^{1,2})-\frac{1}{d}$

It is evident from Table 3 that there are many common elements in the portfolio characteristic formulae for the minimum variance and the growth optimal portfolios. This is not particularly surprising as both portfolios are Markowitz mean- variance efficient portfolios and it is well known that every mean-variance efficient portfolio can be generated as a linear combination of two mean-variance efficient portfolios.

The differences between the characteristics of the growth optimal portfolio and its equivalent minimum variance portfolio have both simple and symmetric structure. Take, for example, the difference between the expected GOP return, r*, and the expected MVP return, r^o :

$$\begin{aligned} r^o - r^* &= r^T(\mathbf{E}^{1,1}\mathbf{r} + \mathbf{E}^{1,2}) - r^T\mathbf{E}^{1,2} \\ &= r^T\mathbf{E}^{1,1}\mathbf{r} \geq \mathbf{0} \end{aligned} \tag{27}$$

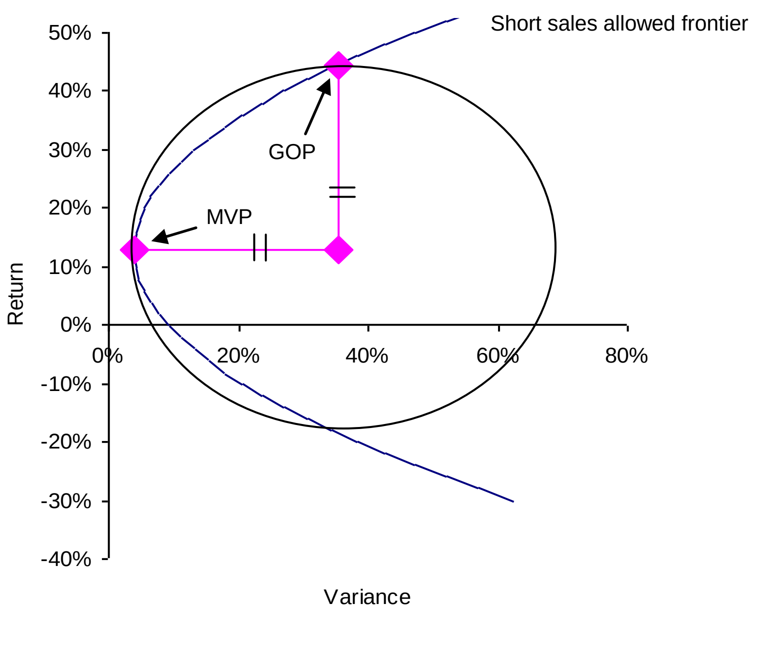
The expected return on the growth optimal portfolio always exceeds that of the minimum variance portfolio as $\mathbf{E}^{11}$ is positive semi-definite.

The gap between the GOP and MVP variances is exactly the same as that between the respective expected returns.

$$\begin{aligned} \sigma^o - \sigma^* &= r^T\mathbf{E}^{1,1}\mathbf{r} - \frac{1}{d} + \frac{1}{d} \\ &= r^T\mathbf{E}^{1,1}\mathbf{r} \geq \mathbf{0} \end{aligned} \tag{28}$$

The equality of the excess return and variance is illustrated in Figure 6, which shows that both the MVP and GOP lie on a circle of radius $r^T\mathbf{E}^{1,1}\mathbf{r}$

Figure 6: MVP and GOP Circle



As both the differences between the expected returns and the variances of the growth optimal and minimum variance portfolios are given by the quadratic expression $r^T\mathbf{E}^{1,1}\mathbf{r}$, the difference in the growth rates of the two portfolios are thus equal to half of this.

$$g^{o} - g^{*} = r^{T}(\frac{1}{2}\mathbf{E}^{1,1}\,\mathbf{r} + \mathbf{E}^{1,2}) - \frac{1}{d} + \frac{1}{d}$$
$$= \frac{1}{2} r^{T}\mathbf{E}^{1,1}\,\mathbf{r} \geq \mathbf{0} \tag{29}$$

The Inclusion of a Risk-Free Asset

Much of the theoretical development of the discipline of finance has relied upon an assumption that a risk-free asset exists and can be included as an asset amongst other risky assets[9]. A relevant question is: how are the properties of the GOP affected by the inclusion of a risk-free asset?

Let us formally examine inclusion of a riskless asset, having return, r_f, as the additional n+1[th] asset added to a portfolio of n risky assets. The proportion held in the riskless asset, w_{rf}, is:

$$w_{rf} = 1 - \sum_{i=1}^{n} w_i = 1 - \iota^{T}\mathbf{w} \tag{30}$$

where **w** is the vector of n risky asset weights. It is convenient to define a simple transformation of the asset returns by subtracting the riskless rate from the original return:

$$\tilde{r}_i = r_i - r_f \tag{31}$$

Using this transformation, the return on the n+1 asset portfolio is expressed as:

$$\mathbf{r} = \mathbf{w}^{T}\,\tilde{\mathbf{r}} + r_f \tag{32}$$

In order to maximise n+1 asset growth subject to portfolio return, r_p, equalling k, we define the Lagrangean expression:

$$L = \mathbf{w}^{T}\,\tilde{\mathbf{r}} + r_f - \tfrac{1}{2}\mathbf{w}^{T}\boldsymbol{\Omega}\,\mathbf{w} - \lambda(\mathbf{w}^{T}\,\tilde{\mathbf{r}} - \tilde{r}_k) \tag{33}$$

where $\tilde{r}_k = k - r_f$. The solution to this problem is:

$$\mathbf{w}_k = \frac{\boldsymbol{\Omega}^{-1}\,\tilde{\mathbf{r}}\,\tilde{r}_k}{\tilde{\mathbf{r}}^{T}\,\boldsymbol{\Omega}^{-1}\,\tilde{\mathbf{r}}} = \left(\frac{k - r_f}{e}\right)\boldsymbol{\Omega}^{-1}\,\tilde{\mathbf{r}} \tag{34}$$

[9] The existence of a riskless asset is problematical. It is the author's view that if one can accept that a riskless asset exists, one will almost certainly accept the more credible proposition that the tooth fairy is alive and enjoys a full life at the bottom of the garden.

where . $e = \tilde{\mathbf{r}}^T \mathbf{\Omega}^{-1} \tilde{\mathbf{r}}$ We shall show later that $\Omega^{-1}\tilde{\mathbf{r}}$ is the n+1 asset absolute growth portfolio. Hence the n+1 asset weights are a linear function of the GOP weights and the portfolio expected return k:

$$\mathbf{w}_k = \left(\frac{k - r_f}{e}\right) \mathbf{w}_{GOP} \tag{35}$$

It is obvious from equation (35) that the weights are a linear function of the portfolio return k. However, as expected the variance and growth rate associated with $\mathbf{w}_k$ are quadratic functions of k:

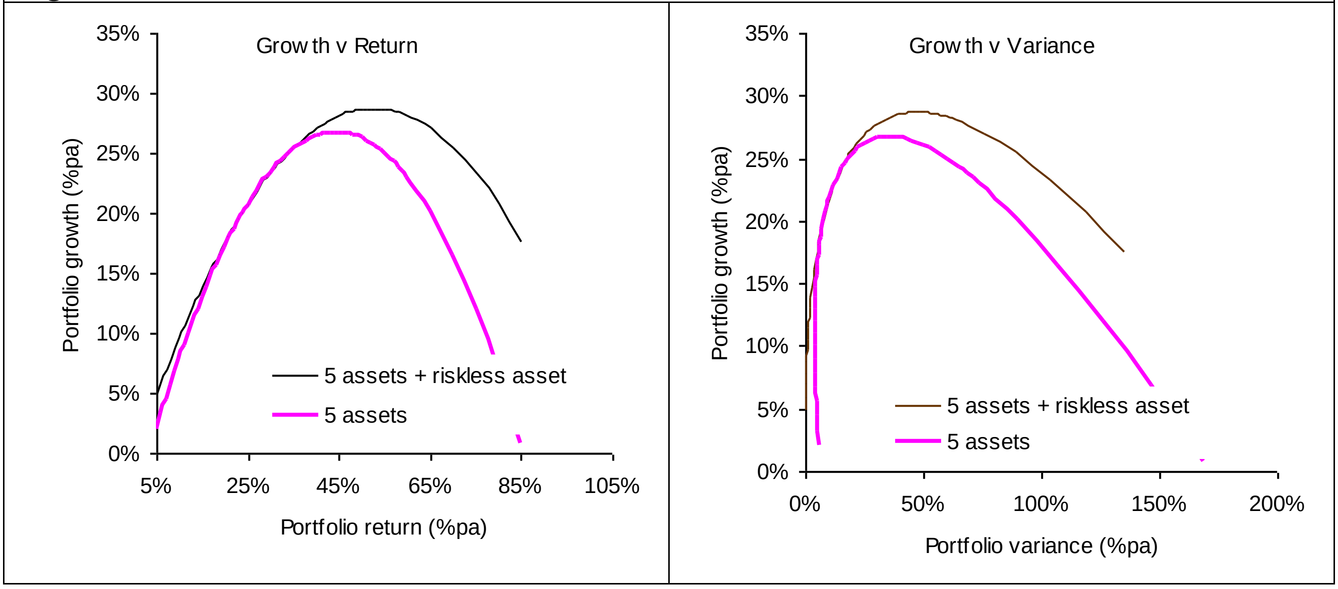
$$\sigma_k^2 = \frac{\tilde{r}_k^2}{e} = \frac{1}{e}(k - r_f)^2 \tag{36}$$

and

$$g_k = r_p - \tfrac{1}{2}\sigma_p^2 = k - \frac{1}{2e}(k - r_f)^2 \tag{37}$$

Figure 7 contrasts, for the application data, the n asset, short sales allowed, risky asset, portfolio (5 assets) with the n+1 asset portfolio of 5 risky assets and a risk-free asset yielding 5%pa. Note that there will always be a point of coincidence between the two portfolios with and without the risk-free asset. This point occurs when the riskless asset naturally takes a weight of zero in the n+1 asset portfolio.

Figure 7: Inclusion of a Risk-Free Asset



It is apparent from equation (35) that the GOP is identified when the first term on the RHS of equation (35) takes the value unity, that is when:

$$k = e + r_f \tag{38}$$

Thus the GOP value of k2 = e + r_f = 48.48%+5%=52.48% in the case of the application data.

The Growth Optimal Pricing Formula

The rate of growth of a portfolio consisting of n risky assets and one riskless asset can be set out as :

$$g = \mathbf{w}^T \tilde{\mathbf{r}} + r_f - \tfrac{1}{2}\mathbf{w}^T \mathbf{\Omega}\,\mathbf{w} \tag{39}$$

as $\dfrac{\delta L}{\delta \mathbf{w}}^T = \tilde{\mathbf{r}} - \mathbf{\Omega}\,\mathbf{w}$

The GOP weights for the n risky assets are:

$$\mathbf{w}^* = \mathbf{\Omega}^{-1}\tilde{\mathbf{r}} \;\text{ and }\; w^*_{rf} = 1 - \sum_{i=1}^{n} w_i \tag{40}$$

Long [1990] highlighted the role of an n+1 asset GOP (includes a riskless asset) as a reference portfolio for the individual assets. That is, it is possible to use the GOP as a numeraire portfolio to “price” the individual assets.

The nx1 vector of covariances of the individual asset’s return with that of the GOP, $\boldsymbol{\sigma}_{i,GOP}$ is:

$$\begin{aligned} \boldsymbol{\sigma}_{i,GOP} &= \mathbf{\Omega}\,\mathbf{w}^* = \mathbf{\Omega}\,\mathbf{\Omega}^{-1}\tilde{\mathbf{r}} \\ &= \tilde{\mathbf{r}} \end{aligned} \tag{41}$$

or in terms of individual assets:

$$\sigma_{i,GOP} = r_i - r_f \tag{42}$$

Equation (41) shows that an asset’s excess return over that of the riskless asset is equal to the covariance of the asset’s return with the return on the GOP. Moreover, there is a relationship between an asset’s excess return and the GOP excess return. We can show that this relationship parallels the CAPM equations that relate asset excess return to the market excess return.

Let us define a GOP beta vector as:

$$\boldsymbol{\beta}_i = \frac{\boldsymbol{\sigma}_{i,GOP}}{\sigma^2_{GOP}} = \frac{\tilde{\mathbf{r}}}{\sigma^2_{GOP}} \tag{43}$$

Now

$$\begin{aligned} \sigma^2_{GOP} &= \mathbf{w}^{*T}\,\mathbf{\Omega}\,\mathbf{w}^* = \tilde{\mathbf{r}}^{\,T}\mathbf{\Omega}^{-1}\tilde{\mathbf{r}} \\ &= r_{GOP} - r_f \end{aligned} \tag{44}$$

as

$$r_{GOP} = \mathbf{w}^{*T}\,\tilde{\mathbf{r}} + r_f = \tilde{\mathbf{r}}^{\,T}\mathbf{\Omega}^{-1}\tilde{\mathbf{r}} + r_f \tag{45}$$

Rearranging equation (43) produces

$$\tilde{\mathbf{r}} = \boldsymbol{\beta}(r_{GOP} - r_f) \tag{46}$$

or in terms of the individual assets

$$r_i - r_f = \beta_i (r_{GOP} - r_f) \tag{47}$$

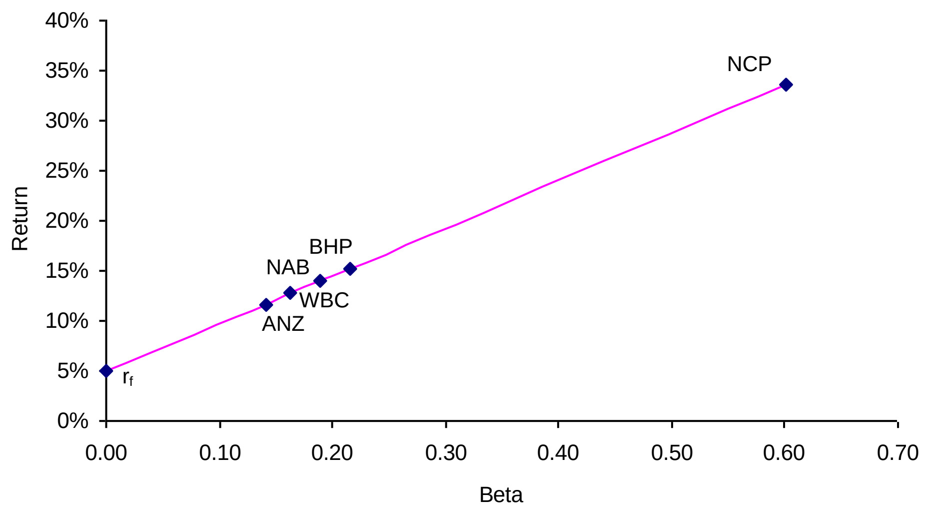
The short sales allowed weights of the assets in the GOP and the GOP betas for the application data are set out in Table 4[10]

Table 4: GOP weights and betas

	Return	Variance	Growth	GOP Weights	beta
ANZ	14.0%	6.3%	10.8%	82.6%	0.19
WBC	11.7%	6.0%	8.7%	-156.1%	0.14
NAB	15.3%	5.5%	12.5%	147.3%	0.22
NCP	33.5%	21.3%	22.9%	123.0%	0.60
BHP	12.7%	6.7%	9.4%	3.7%	0.16
Riskless	5.0%	0.0%	5.0%	-100.5%	0.00
GOP	52.48%	47.48%	28.74%		1.00

Figure 8 illustrates the linear relationship existing between asset return and asset GOP beta.

Figure 8: Asset Return v GOP beta



[10] As the short sales not allowed GOP consists of 100% NCP, we have not included the short sales not allowed GOP equivalent figures as results have little meaning.

Summary

Expected rate of growth, like expected return and variance of return, is a significant characteristic of any investment portfolio. This paper has examined in detail the structure and nature of growth oriented portfolios. The analysis of the growth properties of investment portfolios was undertaken within the Markowitz minimum risk framework, and growth "efficient" portfolios were shown to be also mean-variance efficient portfolios. The structure of the growth optimal portfolio was compared and contrasted with its equivalent minimum variance portfolio. Finally, we derived a growth version of CAPM. Asset expected return was shown to be a linear function of growth beta.

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Appendix

ONE

Growth Application Data

Price data on five Australian companies are employed to illustrate some of the properties of growth portfolios. The companies are Australian and New Zealand Banking Ltd. (ANZ), Westpac Banking Corporation (WBC), National Australia Bank (NAB), News Corporation (NCP) and BHP Billiton (BHP). These companies are the five largest (by market capitalisation) Australian companies that continually traded over the data period of June 1980 to September 2001. Return, volatility and correlation structures for the five companies are set out in table below.

Australian Company Return Statistics					
Returns (%pa)					
	ANZ	*WBC*	*NAB*	*NCP*	*BHP*
Implied μ	13.97%	11.67%	15.25%	33.53%	12.70%
Growth	10.82%	8.69%	12.51%	22.88%	9.36%
Volatility (standard deviation %pa)					
	ANZ	WBC	NAB	NCP	BHP
	25.14%	24.43%	23.41%	46.15%	25.86%
Covariance (%pa)					
	ANZ	*WBC*	*NAB*	*NCP*	*BHP*
ANZ	6.23%	4.50%	3.84%	4.03%	2.81%
WBC	4.50%	5.90%	4.09%	4.80%	2.78%
NAB	3.84%	4.09%	5.35%	4.32%	2.45%
NCP	4.03%	4.80%	4.32%	20.77%	4.86%
BHP	2.81%	2.78%	2.45%	4.86%	6.47%
Correlation					
	ANZ	*WBC*	*NAB*	*NCP*	*BHP*
ANZ	1.00	0.75	0.68	0.35	0.42
WBC	0.75	1.00	0.73	0.43	0.44
NAB	0.68	0.73	1.00	0.40	0.43
NCP	0.35	0.43	0.40	1.00	0.41
BHP	0.42	0.44	0.43	0.41	1.00
The statistics have been estimated from monthly data (June 1980 – September 2001) obtained from the Beacon service supplied by Reuters Australia					

TWO

Positive semi-definiteness of $\mathbf{A}^{1,1}$ requires that for any vector, $\mathbf{a}$:

$$\mathbf{a}^{\mathrm{T}}\left[\boldsymbol{\Omega}^{-1}\left(\mathbf{I}-\frac{\boldsymbol{\iota}^{\mathrm{T}}\boldsymbol{\iota}\boldsymbol{\Omega}^{-1}}{\boldsymbol{\iota}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\boldsymbol{\iota}}\right)\right]\mathbf{a} \geq \mathbf{0}$$

$$\mathbf{a}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{a} - \frac{(\mathbf{a}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\boldsymbol{\iota})^2}{\boldsymbol{\iota}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\boldsymbol{\iota}} \geq \mathbf{0}$$

As $\boldsymbol{\Omega}$ is positive definite so is its inverse $\boldsymbol{\Omega}^{-1}$ and thus it is suitable as a covariance matrix. Choose a vector of random variables, $\mathbf{y}$, such that it has covariance matrix $\boldsymbol{\Omega}^{-1}$. Consider two linear combinations of $\mathbf{y}$, $\mathbf{a}^{\mathrm{T}}\,\mathbf{y}$ and ι^{T}. Thus:

$$\mathrm{Var}\ \left(\mathbf{a}^{\mathrm{T}}\mathbf{y}\right) = \mathbf{a}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\mathbf{a}$$
$$\mathrm{Var}\ \left(\boldsymbol{\iota}^{\mathrm{T}}\mathbf{y}\right) = \boldsymbol{\iota}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\boldsymbol{\iota}$$
$$\mathrm{Co\,var}\left(\mathbf{a}^{\mathrm{T}}\mathbf{y}, \boldsymbol{\iota}^{\mathrm{T}}\mathbf{y}\right) = \mathbf{a}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\boldsymbol{\iota}$$

The Cauchy-Swartz inequality states that:

$$\mathrm{Co\,var}(\mathrm{X},\mathrm{Y})^2 \ \leq \mathrm{Var}(\mathrm{X}) * \mathrm{Var}(\mathrm{Y})$$

Thus it follows:

$$(\mathbf{a}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\boldsymbol{\iota})^2 \ \leq (\mathbf{a}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\,\mathbf{a}) * \boldsymbol{\iota}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\boldsymbol{\iota}$$

and thus:

$$\mathbf{a}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\,\mathbf{a} - \frac{(\mathbf{a}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\boldsymbol{\iota})^2}{\boldsymbol{\iota}^{\mathrm{T}}\boldsymbol{\Omega}^{-1}\boldsymbol{\iota}} \ \geq \mathbf{0}$$

THREE

The growth rate g* of the maximum growth portfolio can be expressed as a function of the corresponding weight vector **w***:

$$\mathrm{g}^* = \mathbf{w}^{*\mathrm{T}}\ \mathbf{r} - \tfrac{1}{2}\,\mathbf{w}^{*\mathrm{T}}\ \boldsymbol{\Omega}\,\mathbf{w}^*$$

where $\mathbf{w^*} = \mathbf{E}^{1,1}\ \mathbf{r} + \mathbf{E}^{1,2}$.

Thus:

$$\begin{aligned}\mathrm{g}^* &= (\mathbf{E}^{1,1}\ \mathbf{r} + \mathbf{E}^{1,2})^{\mathrm{T}}\,\mathbf{r} - \tfrac{1}{2}(\mathbf{E}^{1,1}\ \mathbf{r} + \mathbf{E}^{1,2})^{\mathrm{T}}\,\boldsymbol{\Omega}\,(\mathbf{E}^{1,1}\ \mathbf{r} + \mathbf{E}^{1,2}) \\ &= \mathbf{r}^{\mathrm{T}}\left[\mathbf{E}^{1,1}\left(\mathrm{I} - \tfrac{1}{2}\boldsymbol{\Omega}\mathbf{E}^{1,1}\right)\right]\mathbf{r} + \mathbf{E}^{1,2\,\mathrm{T}}\left(\mathrm{I} - \boldsymbol{\Omega}\mathbf{E}^{1,1}\right)\ \mathbf{r}\ \text{-}\ \mathbf{E}^{1,2\,\mathrm{T}}\boldsymbol{\Omega}\mathbf{E}^{1,2}\end{aligned}$$

differentiating g* wrt r:

$$\left(\frac{\delta \mathrm{g}^*}{\delta \mathrm{r}}\right)^{\mathrm{T}} = 2\left[\mathbf{E}^{1,1}\left(\mathrm{I} - \tfrac{1}{2}\boldsymbol{\Omega}\mathbf{E}^{1,1}\right)\right]\mathbf{r} + \left(\mathrm{I} - \boldsymbol{\Omega}\mathbf{E}^{1,1}\right)\mathbf{E}^{1,2}$$

recognising that $\boldsymbol{\iota}^{\mathrm{T}}\mathbf{E}^{1,1} = \mathbf{0}$, $\boldsymbol{\iota}^{\mathrm{T}}\mathbf{E}^{1,2} = 1$ and $\mathbf{E}^{1,1}\ \boldsymbol{\Omega} = (\mathbf{I}\ \text{-}\ \mathbf{E}^{1,2}\boldsymbol{\iota}^{\mathrm{T}})$ one can simplify the derivative to:

$$\left(\frac{\delta \mathrm{g}^*}{\delta \mathrm{r}}\right)^{\mathrm{T}} = \mathbf{E}^{1,1}\ \ \mathbf{r} + \mathbf{E}^{1,2} = \mathbf{w}^*$$

FOUR

The derivation of the n+1 asset portfolio, including a riskless asset, having a return of k and maximum possible growth, proceeds as follows. We define the Lagreangean expression:

$$L = \mathbf{w}^T \tilde{\mathbf{r}} + r_f - \tfrac{1}{2}\mathbf{w}^T \boldsymbol{\Omega}\,\mathbf{w} - \lambda(\mathbf{w}^T \tilde{\mathbf{r}} - \tilde{r}_k)$$

where r_f is the risk free rate, $\tilde{r}_k = k - r_f$, $\tilde{r}_i = r_i - r_f$ i=1,n. Differentiating the Lagreangean produces:

$$\frac{\delta L}{\delta \mathbf{w}}^T = \tilde{\mathbf{r}} - \boldsymbol{\Omega}\mathbf{w} - \lambda\tilde{\mathbf{r}} = 0$$

$$\frac{\delta L}{\delta \lambda}^T = -(\mathbf{w}^T \tilde{\mathbf{r}} - \tilde{r}_k) = 0$$

The above equations can be expressed in partitioned matrix form as:

$$\begin{pmatrix} \boldsymbol{\Omega} & \tilde{\mathbf{r}} \\ \tilde{\mathbf{r}}^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{w} \\ \lambda \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{r}} \\ \tilde{\mathbf{r}}_k \end{pmatrix}$$

The solution $\mathbf{w}_k$ to the above set of equations is:

$$\mathbf{w}_k = \boldsymbol{\Omega}^{-1}\left(\mathbf{I} - \frac{\tilde{\mathbf{r}}\,\tilde{\mathbf{r}}^T\,\boldsymbol{\Omega}^{-1}}{\tilde{\mathbf{r}}^T\,\boldsymbol{\Omega}^{-1}\,\tilde{\mathbf{r}}}\right)\tilde{\mathbf{r}} + \frac{\boldsymbol{\Omega}^{-1}\,\tilde{\mathbf{r}}}{\tilde{\mathbf{r}}^T\,\boldsymbol{\Omega}^{-1}\,\tilde{\mathbf{r}}}\,\tilde{\mathbf{r}}_k$$

$$= \frac{\boldsymbol{\Omega}^{-1}\,\tilde{\mathbf{r}}}{\tilde{\mathbf{r}}^T\,\boldsymbol{\Omega}^{-1}\,\tilde{\mathbf{r}}}\,\tilde{\mathbf{r}}_k$$

as

$$\boldsymbol{\Omega}^{-1}\left(\mathbf{I} - \frac{\tilde{\mathbf{r}}\,\tilde{\mathbf{r}}^T\,\boldsymbol{\Omega}^{-1}}{\tilde{\mathbf{r}}^T\,\boldsymbol{\Omega}^{-1}\,\tilde{\mathbf{r}}}\right)\tilde{\mathbf{r}} = \boldsymbol{\Omega}^{-1}\left(\tilde{\mathbf{r}} - \frac{\tilde{\mathbf{r}}\,\tilde{\mathbf{r}}^T\,\boldsymbol{\Omega}^{-1}\,\tilde{\mathbf{r}}}{\tilde{\mathbf{r}}^T\,\boldsymbol{\Omega}^{-1}\,\tilde{\mathbf{r}}}\right)\tilde{\mathbf{r}}$$

$$= \mathbf{0}$$